Chapitre 19

TD

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Exercice 6

1.

$$S = \{u \in \mathbb{R}^{\mathbb{N}} \mid \forall n \in \mathbb{N}, u_{n+2} = au_{n+1} + bu_n\}$$

— $S \neq \varnothing$ car $0 \in S$.
— Soient $u, v \in S$ et $\lambda, \mu \in \mathbb{R}$. Soit $n \in \mathbb{N}$.

$$\begin{aligned}(\lambda u + \mu v)(n+2) &= \lambda u_{n+2} + \mu v_{n+2} \\ &= \lambda(au_{n+1} + bu_n) + \mu(av_{n+1} + bv_n) \\ &= a(\lambda u_n + \mu v_n) + b(\lambda u + \mu v) \\ &= a(\lambda u + \mu v)(n+1) + b(\lambda u + \mu v)(n)\end{aligned}$$

Donc $(\lambda u + \mu v) \in S$.

2.

$$\begin{aligned}\Phi : S &\longrightarrow \mathbb{R}^2 \\ (x_n) &\longmapsto (x_0, x_1)\end{aligned}$$

Soit $(u, v) \in S^2$ et $(\lambda, \mu) \in \mathbb{R}^2$.

$$\begin{aligned}\Phi(\lambda u + \mu v) &= (\lambda u_0 + \mu v_0, \lambda u_1 + \mu v_1) \\ &= \lambda(u_0, u_1) + \mu(v_0, v_1) \\ &= \lambda\Phi(u) + \mu\Phi(v)\end{aligned}$$

Donc Φ est linéaire.

$$\begin{aligned}\forall n \in \mathbb{N}, \quad a\alpha^{n+1} + b\alpha^n &= \alpha^n(a\alpha + b)\\ &= \alpha^n + \alpha^2 \quad \text{car solution de l'équation caractéristique}\\ &= \alpha^{n+2}\end{aligned}$$

Donc, $(\alpha^n)_{n\in\mathbb{N}} \in S$ et $\Phi(\alpha^n) = (1, \alpha) \neq (0, 0)$ Donc $\dim\big(\text{Im}(\varphi)\big) > 0$.
De même, $(\beta_n)_{n\in\mathbb{N}} \in S$ et $\Phi(\beta^n) = (1, \beta) \neq (0, 0)$.
On a deux vecteurs non colinéaires dans Im Φ donc $\dim(\text{Im } \Phi) = 2$.
Donc Im $\Phi = \mathbb{R}^2$ et donc Φ est bijective.

Exercise 10

$\dim(\text{Ker } f \cap F) \geqslant \dim F - \text{rk } f$

Let $g : \begin{array}{ccc} ? & \longrightarrow & ? \\ x & \longmapsto & ? \end{array}$. We need to find g such that $\text{Ker}(g) = \text{Ker}(f) \cap F$.

$$\begin{aligned} g(x) = 0 \iff x \in \text{Ker}(g) &\iff x \in \text{Ker}(f) \cap F \\ &\iff x \in \text{Ker}(f) \text{ and } x \in F \\ &\iff f(x) = 0 \text{ and } x \in F \end{aligned}$$

So,

$$\begin{aligned} g : F &\longrightarrow E \\ x &\longmapsto f(x) \end{aligned}$$

f is linear, so, g is also linear.

Thus,

$$\begin{aligned} \dim(F) &= \dim(\text{Ker } g) - \dim(\text{Im } f) \\ &= \dim\big(\text{Ker}(f) \cap F\big) - \dim(\text{Im } f) \end{aligned}$$

We know that

$$\text{Im } f \supset \text{Im } g$$

So,

$$\dim(\text{Im } f) \leqslant \dim(\text{Im } g)$$

So,

$$\dim F \leqslant \dim(\text{Ker}(f) \cap F) + \dim(\text{Im}, f)$$

and so

$$\dim\big(\text{Ker}(f) \cap F\big) \geqslant \dim F - \dim(\text{Im } f)$$

Exercice 1

$E = \mathbb{K}_3[X]$ et $\varphi : \begin{array}{ccc} E & \longrightarrow & \mathbb{K}[X] \\ P & \longmapsto & XP' - P \end{array}$

1. Soit $P \in E$.

$$\begin{aligned} \deg(XP') &= 1 + \deg(P) - 1 = \deg(P) \\ \deg(XP' - P) &\leqslant \deg(P) \leqslant 3 \end{aligned}$$

   donc $\varphi(P) \in E$.
   Soient $P, Q \in E$, $\lambda, \mu \in \mathbb{K}$.

$$\begin{aligned} \varphi(\lambda P + \mu Q) &= X(\lambda P + \mu Q)' - (\lambda P + \mu Q) \\ &= X(\lambda P' + \mu Q') - \lambda P - \mu Q \\ &= \lambda(XP' - P) + \mu(XQ' - Q) \\ &= \lambda\varphi(P) + \mu\varphi(Q) \end{aligned}$$

   Donc $\varphi \in \mathscr{L}(E)$.

2. Soit $P \in E$. On pose $P = aX^3 + bX^2 + cX + d$ avec $(a, b, c, d) \in \mathbb{K}^4$.

$$\begin{aligned} P \in \operatorname{Ker}\varphi &\iff \varphi(P) = 0 \\ &\iff X(3aX^2 + 2bX + c) - (aX^3 + bX^2 + cX + d) = 0 \\ &\iff 2aX^3 + bX^2 + d = 0 \\ &\iff \begin{cases} a = 0 \\ b = 0 \\ d = 0 \end{cases} \\ &\iff P = cX \\ &\iff P \in \operatorname{Vect}(X) \end{aligned}$$

   $\operatorname{Ker}(\varphi) = \operatorname{Vect}(X)$ et $X \neq 0$ donc (X) est une base de $\operatorname{Ker}\varphi$.

$$\begin{aligned} \operatorname{Im}(\varphi) &= \operatorname{Vect}\big(\varphi(1), \varphi(X), \varphi(X^2), \varphi(X^3)\big) \\ &= \operatorname{Vect}(-1, X^2, 2X^3) \end{aligned}$$

   D'après le théorème du rang,

$$\begin{aligned} \dim\big(\operatorname{Im}(\varphi)\big) &= \dim E - \dim(\operatorname{Ker}\varphi) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

   Donc $(1, X^2, X^3)$ est une base de $\operatorname{Im}(\varphi)$.

Exercice 3

$E = \mathbb{R}^4$ et $f : E \to E$ linéaire.

1. $u = (x, y, z, t) \in E$ donc

$$u = xe_1 + ye_2 + ze_3 + te_4$$

donc

$$\begin{aligned}
f(u) &= f(xe_1 + ye_2 + ze_3 + te_4) \\
&= xf(e_1) + yf(e_2) + zf(e_3) + tf(e_4) \\
&= x(1,1,1,1) + y(1,2,1,1) + z(2,3,2,2) + t(-1,-2,-1,-1) \\
&= (x+y+2z-t, x+2y+3z-2t, x+y+2z-t, x+y+2z-t)
\end{aligned}$$

Remarque:

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 1 & 2 & 3 & -2 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x+y+2z-t \\ x+2y+3z-2t \\ x+y+2z-t \\ x+y+2z-t \end{pmatrix}$$

2. Soit $u = (x, y, z, t) \in E$.

$$\begin{aligned}
u \in \operatorname{Ker}(f) &\iff f(u) = (0,0,0,0) \\
&\iff \begin{cases} x+y+2z-t=0 \\ x+2y+3z-2t=0 \\ x+y+2z-t=0 \\ x+y+2z-t=0 \end{cases} \\
&\iff \begin{cases} x+2y+3z-2t=0 \\ x+y+2z-t=0 \end{cases} \\
\underset{L_2 \leftarrow L_2 - L_1}{\iff} & \begin{cases} \boxed{x}+y+2z-t=0 \\ \boxed{y}+z-t=0 \end{cases} \\
&\iff \begin{cases} x=-z \\ y=-z+t \end{cases} \\
&\iff u = (-z, -z+t, z, t) \\
&\qquad = z\underbrace{(-1,-1,1,0)}_{v_1} + t\underbrace{(0,1,0,1)}_{v_2} \\
&\iff u \in \operatorname{Vect}(v_1, v_2)
\end{aligned}$$

Or, (v_1, v_2) est libre donc c'est une base de $\operatorname{Ker}(f)$.

$$\operatorname{Im}(f) = \operatorname{Vect}\big(f(e_1), f(e_2), f(e_3), f(e_4)\big)$$

Or, $\operatorname{rg}(f) = 4 - 2 = 2$
Donc $\big(f(e_1), f(e_2)\big)$ est libre dans $\operatorname{Im}(f)$ donc c'est une base de $\operatorname{Im}(f)$

3. $\operatorname{Ker}(f) \neq \{0\}$ donc f n'est pas injective donc pas bijective.
4.

$$\begin{aligned}
\forall (x,y,z,t) \in E, f((x,y,z,t)) = (&x+y+2z-t, x+2y+3z-2t, \\
&x+y+2z-t, x+y+2z-t)
\end{aligned}$$

Soit $(\alpha, \beta, \gamma, \delta)$.

$$\begin{aligned}
(\alpha, \beta, \gamma, \delta) \in \mathrm{Im}(f) &\iff \exists (x,y,z,t) \in E, (\alpha, \beta, \gamma, \delta) = f\big((x,y,z,t)\big) \\
&\iff \exists (x,y,z,t) \in E, \begin{cases} x + y + 2z - t = \alpha \\ x + 2y + 3z - 2t = \beta \\ x + y + 2z - t = \gamma \\ x + y + 2z - t = \delta \end{cases} \\
&\iff \exists (x,y,z,t) \in E \begin{cases} \boxed{x} + y + 2z - t = \alpha \\ \boxed{y} + z - t = \beta - \alpha \\ 0 = \gamma - \alpha \\ 0 = \delta - \alpha \end{cases} \\
&\iff \begin{cases} \gamma - \alpha = 0 \\ \delta - \alpha = 0 \end{cases} \\
&\iff (\alpha, \beta, \gamma, \delta) = (\alpha, \beta, \alpha, \alpha) \\
&\qquad\qquad\qquad = \alpha(1,0,1,1) + \beta(0,1,0,0)
\end{aligned}$$

Une base de $\mathrm{Im}(f)$ est $\big((1,0,1,1),(0,1,0,0)\big)$.

Exercise 7

—

$$\forall f \in \mathscr{L}(E), \begin{cases} f \circ p \in \mathscr{L}(E) \\ p \circ f \in \mathscr{L}(E) \end{cases}$$

so

$$\frac{1}{2}(f \circ p + p \circ f) \in \mathscr{L}(E)$$

— Let $(f,g) \in \mathscr{L}(E)^2$, $(\lambda, \mu) \in \mathbb{K}^2$.

$$\begin{aligned}
\Phi(\lambda f + \mu g) &= \frac{1}{2}\big((\lambda f + \mu g) \circ p + p \circ (\lambda f + \mu g)\big) \quad = \frac{1}{2}(\lambda f \circ p + \mu g \circ p + \lambda p \circ f + \mu p \circ g) \\
&= \lambda \Phi(f) + \mu \Phi(g)
\end{aligned}$$

Thus,

$$\Phi \in \mathscr{L}\big(\mathscr{L}(E)\big)$$

—

$$\begin{aligned}
E &= \mathrm{Ker}(p) \oplus \mathrm{Im}(p) \\
P_{|\,\mathrm{Ker}(p)} &= 0 \\
P_{|\mathrm{Im}(p)} &= \mathrm{id}
\end{aligned}$$

<u>Analysis</u> Let $f \in \mathrm{Ker}(\Phi)$ so

$$\Phi(f) = 0_{\mathscr{L}(E)}$$

thus

$$\forall x \in E, \Phi(f)(x) = 0_E$$

so

$$\forall x \in \mathrm{Ker}(p), \Phi(f)(x) = 0_E$$

so

$$\forall x \in \mathrm{Ker}(p), f(0_E) + p\big(f(x)\big) = 0_E$$

thus

$$\forall x \in \operatorname{Ker}(p), f(x) \in \operatorname{Ker}(p)$$

Let $x \in \operatorname{Im}(p)$

$$\begin{aligned}\Phi(f)(x) = 0_E \text{ so } & f(x) + p\big(f(x)\big) = 0_E \\ \text{so } & p\big(f(x)\big) = -f(x) \\ \text{so } & -f(x) \in \operatorname{Im}(p) \\ \text{so } & f(x) \in \operatorname{Im}(p) \\ \text{so } & p\big(f(x)\big) = f(x) \\ \text{so } & f(x) = 0_E\end{aligned}$$

<u>SYNTHESIS</u> Let $u \in \mathscr{L}(\operatorname{Ker} p)$. We set $f : E \to E$ defined by

$$\forall x \in E, f(x) = u\big(x - p(x)\big)$$

—

$$\forall x \in E, p\big(x - p(x)\big) = p(x) - p(x) = 0_E$$

so f is well-defined.

— Let $(x, y) \in E^2$ and $(\alpha, \beta) \in \mathbb{K}^2$.

$$\begin{aligned}f(\alpha x + \beta y) &= u\big(\alpha x + \beta y - p(\alpha x + \beta y)\big) \\ &= u\big(\alpha x + \beta y - \alpha p(x) - \beta p(y)\big) \\ &= u\big(\alpha\big(x - p(x)\big) + \beta\big(y - p(y)\big)\big) \\ &= \alpha u\big(x - p(x)\big) + \beta u\big(y - p(y)\big) \\ &= \alpha f(x) + \beta f(y)\end{aligned}$$

Thus,

$$f \in \mathscr{L}(E)$$

— Let $x \in E$.

$$\begin{aligned}\Phi(f)(x) &= \frac{1}{2}\big(f\big(p(x)\big) + \underbrace{p\big(f(x)\big)}_{=0_E \text{ car } f(x) \in \operatorname{Ker}(p)}\big) \\ &= \frac{1}{2}\big(p(x) - p\big(p(x)\big)\big) \\ &= \frac{1}{2}u\big(p(x) - p(x)\big) \\ &= 0_E\end{aligned}$$

So,

$$\Phi(f) = 0_{\mathscr{L}(E)}$$

Thus,

$$f \in \operatorname{Ker}\Phi$$

Exercice 9

1. $E^* = \mathscr{L}(E, \mathbb{K})$

2. Soit $i \in [\![1, n]\!]$. Soient $(x, y) \in E^2$ et $(\lambda, \mu) \in \mathbb{K}^2$.

   On pose $x = \sum_{j=1}^{n} x_j e_j$ avec $(x_1, \ldots, x_n) \in \mathbb{K}^n$ et $y = \sum_{j=1}^{n} y_j e_j$ avec $(y_1, \ldots, y_n) \in \mathbb{K}^n$.

   D'où,

$$\begin{aligned}\lambda x + \mu y &= \lambda \sum_{j=1}^{n} x_j e_j + \mu \sum_{j=1}^{n} y_j e_j \\ &= \sum_{j=1}^{n} (\lambda x_j + \mu y_j) e_j\end{aligned}$$

$$\begin{aligned}e_i^*(\lambda x + \mu y) &= \lambda x_i + \mu y_i \\ &= \lambda e_i^*(x) + \mu e_i^*(y)\end{aligned}$$

   Donc, $e_i^* \in E^*$.
   Soit $(\lambda_1, \ldots, \lambda_n) \in \mathbb{K}^n$. On suppose que

$$\sum_{i=1}^{n} \lambda_i e_i^* = 0_{E^*}$$

   Donc

$$\forall x \in E, \sum_{i=1}^{n} \lambda_i e_i^*(x) = 0_{\mathbb{K}}$$

$$\forall j \in [\![1, n]\!], \sum_{i=1}^{n} \lambda_i \underbrace{e_i^*(e_j)}_{\delta_{i,j}} = 0_{\mathbb{K}} \text{ donc } \lambda_j = 0_{\mathbb{K}}$$

   Donc $(e_1^*, \ldots, e_n^*)$ est libre.

3. Soit $f \in E^*$. On va montrer que

$$f = \sum_{i=1}^{n} f(e_i) e_i^*$$

   En effet, pour tout $x \in E$:

$$\begin{aligned}f(x) = f\left(\sum_{i=1}^{n} e_i^*(x) e_i\right) &= \sum_{i=1}^{n} f(e_i) e_i^*(x) \\ &= \left(\sum_{i=1}^{n} f(e_i) e_i^*\right)(x)\end{aligned}$$

   Donc $E^* = \mathrm{Vect}(e_1^*, \ldots, e_n^*)$. Donc $\dim(E^*) = \dim(E)$.

4.

$$\begin{aligned}\Phi(u) : E^* &\longrightarrow \mathbb{K} \\ f &\longmapsto f(u)\end{aligned}$$

   Soient $f, g \in E^*$ et $\lambda, \mu \in \mathbb{K}$.

$$\begin{aligned}\Phi(u)(\lambda f + \mu g) &= (\lambda f + \mu g)(u) \\ &= \lambda f(u) + \mu g(u) \\ &= \lambda \Phi(f) + \mu \Phi(g)\end{aligned}$$

   Donc $\Phi(u) = \mathscr{L}(E^*, \mathbb{K}) = (E^*)^* = E^{**}$.

5.

$$\begin{aligned}\Phi : E &\longrightarrow E^{**}\\ u &\longmapsto \Phi(u)\end{aligned}$$

Soient $(u,v) \in E^2$ et $(\lambda,\mu) \in \mathbb{K}^2$.

$$\Phi(\lambda u + \mu v) : E^* \longrightarrow \mathbb{K}$$
$$\lambda\Phi(u) + \mu\Phi(v) : E^* \longrightarrow \mathbb{K}$$

Soit $f \in E^*$.

$$\begin{aligned}\Phi(\lambda u + \mu v)(f) &= f(\lambda u + \mu v)\\ &= \lambda f(u) + \mu f(v)\\ &= \lambda\Phi(u)(f) + \mu\Phi(v)(f)\\ &= \big(\lambda\Phi(u) + \mu\Phi(v)\big)(f)\end{aligned}$$

Donc $\Phi \in \mathscr{L}(E, E^{**})$.

Soit $u \in E$.

$$\begin{aligned}\Phi(u) = 0_{E^{**}} &\iff \forall f \in E^*, \Phi(u)(f) = 0_{\mathbb{K}}\\ &\iff \forall f \in E^*, f(u) = 0_{\mathbb{K}} \implies \qquad \forall i \in [\![1,n]\!], e_i^*(u) = 0_{\mathbb{K}}\\ &\implies u = \sum_{i=1}^{n} e_i^*(u)e_i = 0_E\end{aligned}$$

Donc, $\operatorname{Ker}\Phi = \{0_E\}$ donc Φ est injective.

6.

$$\dim(E^{**}) = \dim(E^*) = \dim(E)$$

donc

$$\Phi \in \mathrm{GL}(E, E^{**})$$

Exercice 11

1. Soient E un $\mathbb{K}$-espace vectoriel de dimension finie n et $f, g \in \mathscr{L}(E)$.

$$\begin{aligned}f_{|\mathrm{Im}(g)} : \mathrm{Im}(g) &\longrightarrow E\\ x &\longmapsto f(x)\end{aligned}$$

Soit $x \in \mathrm{Im}(g)$.

$$\begin{aligned}x \in \operatorname{Ker}(f_{|\mathrm{Im}(g)}) &\iff f_{|\mathrm{Im}(g)}(x) = 0\\ &\iff \begin{cases} f(x) = 0\\ x \in \mathrm{Im}(g)\end{cases}\\ &\iff x \in \operatorname{Ker}(f) \cap \mathrm{Im}(g)\end{aligned}$$

$$\operatorname{Ker}(f_{|\mathrm{Im}(g)}) = \operatorname{Ker}(f) \cap \mathrm{Im}(g)$$

$$\dim(\text{Im } g) = \dim\big(f(\text{Im } g)\big) + \dim(\text{Ker } f \cap \text{Im } g)$$
$$\dim\big(f(\text{Im } g)\big) = \dim\big(\text{Im}(f \circ g)\big) = \text{rg}(f \circ g)$$
$$\text{rg}(f \circ g) = \text{rg}(g) - \dim(\text{Im } g \cap \text{Ker } f)$$
$$\dim(E) = n = \text{rg}(g) + \dim(\text{Ker } f)$$

donc

$$-n + \text{rg}(f) = \dim(\text{Ker } f) \leqslant - \dim(\text{Im } g \cap \text{Ker } f)$$
$$\text{rg}(f \circ g) \geqslant \text{rg}(g) + \text{rg}(f) - n$$

2. <u>Analyse</u> Soit $f \in \mathscr{L}(E)$ et $\dim(E) = 3$. On suppose $f \circ f = 0$.

$$\text{rg}(0) = \text{rg}(f \circ f) \geqslant \text{rg}(f) = \text{rg}(f) + \text{rg}(f) - n$$

Donc

$$0 \leqslant 2\,\text{rg}(f) - n$$
$$3 = n \geqslant 2\,\text{rg}(f)$$

Donc,

$$\text{rg}(f) \in \{0, 1\}$$

et donc

$$\dim(\text{Im } f) \in \{1, 0\}$$

Soit $u \in \text{Im}(f) \setminus \{0\}$ donc Vect $u = \text{Im } f$.

$$\forall x \in K, \exists \lambda(x) \in \mathbb{K}, f(x) = \lambda(x) \cdot u$$

<u>Synthèse</u>

— $f = 0$
— On suppose $f \neq 0$. Soit $u \in E$, $\lambda \in \mathscr{L}(E, \mathbb{K}) = E^*$.

$$f : E \longrightarrow E$$
$$x \longmapsto \lambda(x) \cdot u$$

Soit $x \in E$.

$$\begin{aligned}(f \circ f)(x) &= f\big(\lambda(x) \cdot u\big) \\ &= \lambda\big(\lambda(x) \cdot u\big) \\ &= \lambda(x)\lambda(u) \cdot u\end{aligned}$$

Soient $(x, y) \in E^2$ et $\alpha, \beta \in \mathbb{K}^2$.

$$f(\alpha a + \beta y) = \lambda(\alpha x + \beta y) \cdot u$$
$$\|$$
$$\alpha f(x) + \beta f(y) = \alpha\lambda(x) \cdot u + \beta\lambda(y) \cdot u$$

Donc,

$$\lambda(\alpha x + \beta y) \cdot u = \big(\alpha\lambda(x) + \beta\lambda(y)\big) \cdot u$$

et donc

$$\lambda(\alpha x + \beta y) = \alpha\lambda(x) + \beta\lambda(y)$$
$$f(u) = u$$

Soit $v \in E$ tel que $f(v) = u$.

$$f(u) = f\big(f(v)\big) = 0 = \lambda(u) \cdot u$$

Donc $\lambda(u) = 0$ et donc $u \in \text{Ker}(\lambda)$.

Exercice 12

Partie I.

Soit $N \in \mathbb{N}_*$. On pose

$$\forall P \in \mathbb{R}_N[X], \Delta(P) = P(X+1) - P(X).$$

1. Soit $P \in \mathbb{R}_N[X]$. On pose

$$P = \sum_{k=0}^{N} a_k X^k$$

On pose $N' = \deg P$. Donc,

$$\begin{aligned} P(X+1) &= \sum_{k=0}^{N'} a_k (X+1)^k \\ &= \sum_{k=0}^{N'} a_k \sum_{i=0}^{k} \binom{k}{i} X^i \\ &= \sum_{i=0}^{N'} \left(\sum_{k=i}^{N'} a_k \binom{k}{i} \right) X^i \\ &= \sum_{k=0}^{N'} \left(\sum_{i=k}^{N'} a_i \binom{i}{k} \right) X^k \end{aligned}$$

Donc,

$$\begin{aligned} \Delta(P) = P(X+1) - P(X) &= \sum_{k=0}^{N'} \left(\sum_{i=k}^{N'} a_i \binom{i}{k} \right) X^k - \sum_{k=0}^{N'} a_k X^k \\ &= \sum_{k=0}^{N'} \left(\sum_{i=k}^{N'} a_i \binom{i}{k} - a_k \right) X^k \end{aligned}$$

Pour $k = N'$, $a_{N'} \binom{N'}{N'} - a_{N'} = 0$.
Pour $k = N' - 1$,

$$a_{N'-1} \binom{N'-1}{N'-1} + a_{N'} \binom{N'}{N'-1} - a_{N'-1} \leqslant N' a_{N'} \neq 0 \text{ si } N' \neq 0.$$

Si $\deg P > 0$, alors $\deg \big(\Delta(P)\big) = \deg(P) - 1$.
Si $\deg P \leqslant 0$, alors $\deg \Delta(P) = -\infty$.

2. $\deg \big(\Delta(P)\big) \leqslant \deg(P) \leqslant N$ donc $\Delta(P) \in \mathbb{R}_N[X]$.
Soient $(P, Q) \in \mathbb{R}_N[X]^2$ et $(\lambda, \mu) \in \mathbb{R}^2$.

$$\begin{aligned} \Delta(\lambda P + \mu Q) &= (\lambda P + \mu Q)(X+1) - (\lambda P + \mu Q)(X) \\ &= \lambda P(X+1) + \mu Q(X+1) - \lambda P(X) - \mu Q(X) \\ &= \lambda \Delta(P) + \mu \Delta(Q) \end{aligned}$$

Donc $\Delta \in \mathscr{L}\big(\mathbb{R}_n[X]\big)$.

3.

$$\begin{aligned} P \in \operatorname{Ker}(\Delta) &\iff \Delta(P) = 0 \\ &\iff P(X+1) - P(X) = 0 \\ &\iff \deg(P) \leqslant 0 \\ &\iff P \in \operatorname{Vect}(1) \end{aligned}$$

(1) est libre donc c'est une base.

4. D'après le théorème du rang,

$$\dim\big(\mathbb{R}_N[X]\big) = \dim(\text{Ker}\,\Delta) + \dim(\text{Im}\ \Delta)$$
$$\dim(\text{Im}\ \Delta) = N$$

donc

$$\text{Im}\ \Delta \neq \mathbb{R}_N[X]$$

donc Δ n'est pas surjective.

Partie II.

1. Soit $n \in \mathbb{N}_*$.

$$\begin{aligned}
\Delta(P_n) &= P_n(X+1) - P_n(X) \\
&= \frac{1}{n!}\prod_{k=0}^{n-1}(X+1-k) - \frac{1}{n!}\prod_{k=0}^{n-1}(X-k) \\
&= \frac{1}{n!}\left(\prod_{k=0}^{n-1}(X+1-k) - \prod_{k=0}^{n-1}(X-k)\right) \\
&= \frac{1}{n!}\left(\prod_{i=-1}^{n-2}(X-i) - \prod_{k=0}^{n-1}(X-k)\right) \\
&= \frac{1}{n!}\left(\prod_{i=0}^{n-2}(X-i)\big((\cancel{X}+\cancel{1}) - (\cancel{X}-n+\cancel{1})\big)\right) \\
&= \frac{1}{(n-1)!}\prod_{i=0}^{n-2}(X-i) \\
&= P_{n-1}
\end{aligned}$$

Si $n = 0$, $P_n = 1$ donc $\Delta(P_0) = 0$.

2.

$$\forall i \neq j, \deg(P_i) \neq \deg(P_j)$$

donc $(P_0, \dots, P_n)$ est libre.
Or, il y a $n+1$ vecteurs et

$$\dim\big(\mathbb{R}_n[X]\big) = n+1$$

donc $(P_0, \dots, P_N)$ est une base de $\mathbb{R}_N[X]$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1/2 & 1/3 \\ 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 0 & 1/6 \end{pmatrix}$$

car

$$\begin{aligned}
P_0 &= 1 \\
P_1 &= X \\
P_2 &= \frac{1}{2}X^2 - \frac{1}{2}X \\
P_3 &= -\frac{1}{6}(X^3 - 3X^2 + 2X)
\end{aligned}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1/2 & 1/3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/6 & 0 & 0 & 0 & 1 \end{array}\right) \underset{\substack{L_2 \leftarrow L_2 + L_3 \\ L_3 \leftarrow 2L_3 \\ L_4 \leftarrow 6L_4}}{\sim} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1/6 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 6 \end{array}\right)$$
$$\underset{\substack{L_3 \leftarrow L_3 + L_4 \\ L_2 \leftarrow L_2 + \frac{1}{6}L_4}}{\sim} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 6 \end{array}\right)$$

Donc,

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

On a

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 6 \\ 6 \end{pmatrix}$$

Donc $X^3 = P_1 + 6P_2 + 6P_3$. Donc, $Q = P_2 + 6P_3 + 6P_4$ et aussi $\Delta(P) = X^3$. donc

$$\forall n \in \mathbb{N}, \sum_{k=0}^{n} k^3 = \sum_{k=0}^{n} \big(Q(n+k) - Q(k)\big)$$

Donc

$$\forall n \in \mathbb{N}, \sum_{k=0}^{n} k^3 = Q(n+1) - Q(0)$$