— *Homework* —

Semantics and Verification

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1 Toward Stone Duality.

Question 1. *Show that every Stone space (X,Ω) is Hausdorff (if $x,y \in X$ are distinct, there there are disjoint $U,V \in \Omega$ such that $x \in U$ and $y \in V$).*

Let $x,y \in X$ be two distinct points of a Stone space (X,Ω). As, (X,Ω) is T_0 and without loss of generality, there exists $W \in \Omega$ such that $x \in W$ and $y \notin W$. As (X,Ω) is zero-dimensional, we can write $W =: \bigcup_{i\in I} W_i$ where $W_i \in \mathbf{K}\Omega$ for every $i \in I$. Thus, there exists a clopen set $U := W_i \in \Omega$ such that $x \in W_i \subseteq U$. Define $V := X \setminus U \in \Omega$, and we have that $x \in U$, $y \in V$ (as $y \notin W \supseteq U$) and the open sets U and V are disjoint. We can conclude that every Stone space is Hausdorff.

Question 2. *Show that $\le$ is a partial order on $\mathfrak{L}(\mathsf{LML})$.*

We start by showing the following lemma.

Lemma 1. We have $\phi \le \psi$ if and only if $[\![\phi]\!] \subseteq [\![\psi]\!]$.

Proof. We have that $\phi \le \psi$ iff $\phi \equiv \phi \wedge \psi$ iff $[\![\phi]\!] = [\![\phi \wedge \psi]\!] = [\![\phi]\!] \cap [\![\psi]\!]$ (that last equality is by definition of $[\![-]\!]$) iff $[\![\phi]\!] \subseteq [\![\psi]\!]$. □

We can thus easily show that $\le$ is a partial order.

- ▷ *Reflexivity*. As $[\![\phi]\!] \subseteq [\![\phi]\!]$, we have that $\phi \le \phi$ for every $\phi \in \mathfrak{L}(\mathsf{LML})$.
- ▷ *Transitivity*. For any $\phi,\psi,\vartheta \in \mathfrak{L}(\mathsf{LML})$, if $\phi \le \psi$ and $\psi \le \vartheta$ then, by the lemma, $[\![\phi]\!] \subseteq [\![\psi]\!] \subseteq [\![\vartheta]\!]$, thus we have $[\![\phi]\!] \subseteq [\![\vartheta]\!]$, *i.e.* $\phi \le \vartheta$.
- ▷ *Antisymmetry*. For any $\phi,\psi \in \mathfrak{L}(\mathsf{LML})$, if $\phi \le \psi$ and $\psi \le \phi$ then, by double inclusion with the above lemma, $[\![\phi]\!] = [\![\psi]\!]$ thus $\phi = \psi$ as we consider LML-formulae quotiented by $\equiv$.

2 Lattices and Boolean Algebras.

2.1 Semilattices.

Question 3. *Let $(L, \leq)$ be a partial order.*

1. *Show that $(L, \leq)$ is a meet semilattice if, and only if, L has binary meets $\wedge : L \times L \to L$ and greatest element $\top \in L$.*
2. *Show that $(L, \leq)$ is a join semilattice if, and only if, L has binary joins $\vee : L \times L \to L$ and least element $\bot \in L$.*

1. If $(L, \leq)$ is a meet semilattice, then L has binary meets and a greatest element $\top = \bigwedge \emptyset$ (any element is a lower bound of $\emptyset$, thus the greatest lower bound of $\emptyset$ is the greatest element).

Now, suppose $(L, \leq)$ has a binary meet $\wedge$ and a greatest element $\top$. Consider $\{a_i \mid i \in I\}$ a finite subset of elements of L. By induction on $\#I \in \mathbb{N}$, we define $\bigwedge_{i \in I} a_i \in I$ and show that $\bigwedge_{i \in I} a_i$ is a meet of the finite set $\{a_i \mid i \in I\}$ (like the notation suggests).

- ▷ Define $\bigwedge_{i \in \emptyset} a_i := \top \in L$; as any element is a lower bound of $\emptyset$, the greatest lower bound of $\emptyset$ is the greatest element.
- ▷ Consider $I := J \sqcup \{i\}$. By induction hypothesis, we have that $\bigwedge_{j \in J} a_j$ exists in L and is a meet of $\{a_j \mid j \in J\}$ in $(L, \leq)$. Define

  $$\bigwedge_{k \in I} a_k := \Big(\bigwedge_{j \in J} a_j\Big) \wedge a_i \in I.$$

  We have that $\bigwedge_{k \in I} a_k$ is a lower bound of $\{a_k \mid k \in I\}$. Consider an element a_k with $k \in I$. If $k \in J$ then $a_k \leq \bigwedge_{j \in J} a_j \leq \bigwedge_{k' \in I} a_{k'}$. Otherwise $k = i$ and we immediacy have that $a_i \leq \bigwedge_{k' \in I} a_{k'}$.

  Consider a lower bound $b \in L$ of $\{a_k \mid k \in I\}$, then b is a lower bound of $\{a_j \mid j \in J\}$ and $b \leq a_i$. We have $b \leq \bigwedge_{j \in J} a_j$ and $b \leq a_i$, therefore $b \leq \bigwedge_{k \in I} a_k$.

  We can conclude that $\bigwedge_{k \in I} a_k$ is a meet of $\{a_k \mid k \in K\}$.

Finally, we have that $(L, \leq)$ has finite meets.

2. This results follows from 1 when considering the partial order $(L, \geq)$, by duality. Meets in $(L, \geq)$ are exactly joins in $(L, \leq)$, and the greatest element of $(L, \geq)$ is the least element of $(L, \leq)$, and *vice versa*.

> **Note.** In the following, when I will be dealing with multiple partial orders on the same set (*e.g.* $\leq$ and $\geq$), I will write $\bigwedge_\leq$ for the meet operator in poset $(I, \leq)$, $\bigvee_\leq$ for the join operator in poset $(I, \leq)$, $\top_\leq$ for the greatest element in poset $(I, \leq)$ and $\bot_\leq$ for the least element in poset $(I, \leq)$.

Question 4. *Prove the following.*

1. *Let $(L, \leq)$ be a meet semilattice with binary meets $\wedge : L \times L \to L$ and greatest element $\top \in L$. Then $(L, \wedge, \top)$ is a commutative mooned in which every element is idempotent. Moreover, we have $a \leq b$ iff $a = a \wedge b$.*
2. *Let $(L, \leq)$ be a join semilattice with binary joins $\vee : L \times L \to L$ and least element $\bot$. Then $(L, \vee, \bot)$ is a commutative mooned in which every element is idempotent. Moreover, we have $a \leq b$ iff $b = a \wedge b$.*

1. Let $a, b, c \in L$. First, we have that $a \wedge b = \bigwedge\{a, b\} = \bigwedge\{b, a\} = b \wedge a$ thus the binary meet operation $\wedge$ is commutative. Then, as a special case of the previous question, we have that a and $\top \wedge a$ are both meets of $\{a\}$. And, by unicity of meets (*i.e.* antisymmetry of $\leq$, mainly), they are equals. Also as a special case of the previous question, we have that elements
$$a \wedge (b \wedge c) = \top \wedge (a \wedge (b \wedge c))$$
and
$$(a \wedge b) \wedge c = c \wedge (a \wedge b) = \top \wedge (c \wedge (a \wedge b))$$
are both meets of the set $\{a, b, c\}$, thus are equal. Next, we have that
$$a \wedge a = \bigwedge\{a, a\} = \bigwedge\{a\} = \top \wedge a = a$$
(penultimate equality is from last question), thus $a \wedge a = a$. Finally, we have that:
   - ▷ if $a = a \wedge b$ then a is a lower bound of $\{a, b\}$ thus $a \leq b$;

- ▷ if $a \leq b$ then $a = a \wedge b$ as a is a lower bound of $\{a, b\}$ and any lower bound c of $\{a, b\}$ must satisfy $c \leq a$.

2. Consider the meet semilattice $(L, \geq)$ and apply the results above. Meets in $(L, \geq)$ are exactly joins in $(L, \leq)$, and the greatest element of $(L, \geq)$ is the least element of $(L, \leq)$, and *vice versa*. The last statement follows from the equivalence:

$$a \leq b \quad \text{iff} \quad b \geq a \quad \text{iff} \quad b = a \wedge_{\geq} b \quad \text{iff} \quad b = a \vee_{\leq} b,$$

where the second "iff" follows from the result above for $(L, \geq)$, and the last one follows from the equality $a \wedge_{\geq} b = a \vee_{\leq} b$.

Question 5. *Prove the following.*

1. *Given a commutative monoid $(L, \wedge, \top)$ in which every element is idempotent, let $a \leq_{\wedge} b$ iff $a = a \wedge b$. Then $(L, \leq_{\wedge})$ is a meet semilattice with binary meets given by $\wedge$ and greatest element $\top$.*
2. *Given a commutative monoid $(L, \vee, \bot)$ in which every element is idempotent, let $a \leq_{\vee} b$ iff $b = a \vee b$. Then $(L, \leq_{\vee})$ is a join semilattice with binary joins given by $\vee$ and least element $\bot$.*

1. Let us start by showing that $(L, \leq_{\wedge})$ is a partial order.

   - ▷ *Reflexivity.* As $a \wedge a = a$ by idempotence, we have $a \leq_{\wedge} a$.
   - ▷ *Antisymmetry.* If $a \leq_{\wedge} b$ and $b \leq_{\wedge} a$ then, by commutativity, we have $a \wedge b = a = b$.
   - ▷ *Transitivity.* If $a \leq_{\wedge} b$ and $b \leq_{\wedge} c$ then, by associativity,

   $$a = a \wedge b = a \wedge (b \wedge c) = (a \wedge b) \wedge c = a \wedge c,$$

   thus $a \leq_{\wedge} c$.

   By question 3, it suffices to show that $(L, \leq_{\wedge})$ that binary meets for poset $(L, \leq_{\wedge})$ are $\wedge$ and that $\top$ is the greatest element of poset $(L, \leq_{\wedge})$. Consider a, b, c three arbitrary elements of L.

   - ▷ For any $b \in L$, we have $b \wedge \top = b$ (as $\top$ is a neutral element) and thus $b \leq_{\wedge} \top$ for all $b \in L$, so $\top$ is the greatest element of $(L, \leq_{\wedge})$.

▷ Firstly, element $a \wedge b$ is a lower bound of $\{a, b\}$ as

$$\begin{aligned} a \wedge b \leq_\wedge a \quad &\text{iff} \quad a \wedge b = (a \wedge b) \wedge a \\ a \wedge b \leq_\wedge b \quad &\text{iff} \quad a \wedge b = (a \wedge b) \wedge b \end{aligned}$$

and the latter equalities are true by idempotence, associativity, and finally commutativity. Secondly, consider $c \in L$ such that we have $c \leq_\wedge a$ and $c \leq_\wedge b$, then $c \wedge a = c = c \wedge b$. We therefore have that $c \leq_\wedge a \wedge b$, as

$$c \wedge (a \wedge b) = (c \wedge a) \wedge b = c \wedge b = c.$$

We can conclude that $\wedge$ is the binary meet operator in $(L, \leq_\wedge)$.

2. Applying the previous result with the commutative monoid $(L, \vee, \bot)$, we obtain that $(L, \geq_\vee)$[a] is a meet semilattice where binary meets for $\geq_\vee$ are given by $\vee$ and the greatest element for $\geq_\vee$ is $\bot$. We can thus conclude that $(L, \leq_\vee)$ is a join semilattice where binary joins for $\leq_\vee$ are given by $\vee$ and the least element for $\leq_\vee$ is $\bot$.

Question 6. *Show the following, for the partial order $(\mathfrak{L}(\mathsf{LML}), \leq)$:*

1. $(\mathfrak{L}(\mathsf{LML}), \leq)$ *is a meet semilattice with greatest element* $\top$ *and binary joins given by*

$$\begin{aligned} - \wedge - : \mathfrak{L}(\mathsf{LML}) \times \mathfrak{L}(\mathsf{LML}) &\longrightarrow \mathfrak{L}(\mathsf{LML}) \\ (\phi, \psi) &\longmapsto \phi \wedge \psi\,; \end{aligned}$$

2. $(\mathfrak{L}(\mathsf{LML}), \leq)$ *is a join semilattice with least element* $\bot$ *and binary joins given by*

$$\begin{aligned} - \vee - : \mathfrak{L}(\mathsf{LML}) \times \mathfrak{L}(\mathsf{LML}) &\longrightarrow \mathfrak{L}(\mathsf{LML}) \\ (\phi, \psi) &\longmapsto \phi \vee \psi. \end{aligned}$$

We will use the lemma proven in question 2 (lemma 1, page 1).

[a] The notation is, in a way, “context-sensitive,” as for an arbitrary monoid $(M, \circledast, \mathbf{I})$, we can either define $a \leq_\circledast b$ as $a \circledast b = a$ or $a \leq_\circledast b$ as $a \circledast b = b$.

1. We only need to show that $- \wedge -$ defines a binary meet for $(\mathfrak{L}(\mathsf{LML}), \leq)$ and that $\top$ is a greatest element.

   For any $\phi \in \mathfrak{L}(\mathsf{LML})$, we have $\phi \leq \top$ as $[\![\phi]\!] \subseteq [\![\top]\!] = (\mathbf{2}^{\mathrm{AP}})^{\omega}$, thus $\top$ is the greatest element.

   For any formulae $\phi, \psi \in \mathfrak{L}(\mathsf{LML})$, we have that $\phi \wedge \psi \leq \phi$ and $\phi \wedge \psi \leq \psi$ as both $[\![\phi]\!]$ and $[\![\psi]\!]$ are supersets of $[\![\phi \wedge \psi]\!] = [\![\phi]\!] \cap [\![\psi]\!]$ (by definition of interpretation $[\![-]\!]$). Then, if $\vartheta \leq \phi$ and $\vartheta \leq \psi$, we have that $[\![\vartheta]\!] \subseteq [\![\phi]\!]$ and $[\![\vartheta]\!] \subseteq [\![\psi]\!]$ thus $[\![\vartheta]\!] \subseteq [\![\phi]\!] \cap [\![\psi]\!] = [\![\phi \wedge \psi]\!]$, therefore $\vartheta \leq \phi \wedge \psi$.

   We can conclude that $(\mathfrak{L}(\mathsf{LML}), \leq)$ is a meet semilattice with greatest element $\top$ and binary meets given by $- \wedge -$.

2. We only need to show that $- \vee -$ defines a binary join for $(\mathfrak{L}(\mathsf{LML}), \leq)$ and that $\bot$ is a least element.

   For any $\phi \in \mathfrak{L}(\mathsf{LML})$, we have $\bot \leq \phi$ as $\varnothing \subseteq [\![\bot]\!] \subseteq [\![\phi]\!]$, thus $\bot$ is the least element.

   For any formulae $\phi, \psi \in \mathfrak{L}(\mathsf{LML})$, we have that $\phi \leq \phi \vee \psi$ and $\psi \leq \phi \vee \psi$ as both $[\![\phi]\!]$ and $[\![\psi]\!]$ are subsets of $[\![\phi \vee \psi]\!] = [\![\phi]\!] \cup [\![\psi]\!]$ (by definition of interpretation $[\![-]\!]$). Then, if $\phi \leq \vartheta$ and $\psi \leq \vartheta$, we have that $[\![\phi]\!] \subseteq [\![\vartheta]\!]$ and $[\![\psi]\!] \subseteq [\![\vartheta]\!]$ thus $[\![\phi \vee \psi]\!] = [\![\phi]\!] \cup [\![\psi]\!] \subseteq [\![\vartheta]\!]$, therefore $\phi \vee \psi \leq \vartheta$.

   We can conclude that $(\mathfrak{L}(\mathsf{LML}), \leq)$ is a join semilattice with least element $\bot$ and binary joins given by $- \vee -$.

Question 7. *Show that a map of meet* (resp. *join*) *semilattices is monotone.*

Let $f : L \to L'$ be an arbitrary function where $(L, \leq)$ and $(L', \leq')$ are partial orders.

1. Suppose $f : (L, \leq) \to (L', \leq')$ is a map of meet semilattices. Let $a, b \in L$. If $a \leq b$, then $a \wedge b = a$ and, as f preserves finite meets,

   $$f(a) \wedge' f(b) = f(a \wedge b) = f(a),$$

   and thus $f(a) \leq' f(b)$. Therefore, f is monotone.

2. Suppose $f : (L, \leq) \to (L', \leq')$ is a map of join semilattices. Let $a, b \in L$. If $a \leq b$, then $a \vee b = b$ and, as f preserves finite joins,

$$f(a) \vee' f(b) = f(a \vee b) = f(b),$$

and thus $f(a) \leq' f(b)$. Therefore, f is monotone.

2.2 Lattices.

Question 8. *Consider the partial order $(L, \sqsubseteq)$ where*

$$L := \mathbb{N} \cup \{\alpha, \beta, \top\},$$

where $\sqsubseteq$ is the reflexive-transitive closure of $\sqsubset$, where

$$a \sqsubset b \quad \textit{iff} \quad \begin{cases} a < b \textit{ in } \mathbb{N} \\ \qquad \textit{or} \\ a \in \mathbb{N} \textit{ and } b \in \{\alpha, \beta\} \\ \qquad \textit{or} \\ a \in \{\alpha, \beta\} \textit{ and } b = \top. \end{cases}$$

Show that $(L, \sqsubseteq)$ is a join semilattice but is not a lattice.

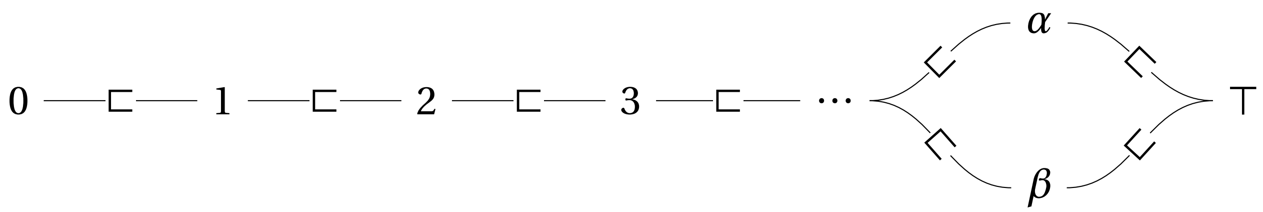


Figure 1 | *Hasse diagram of $(L, \sqsubseteq)$ from question 8*

Note: Hasse diagrams are usually read bottom-to-top, but this one is drawn left-to-right for convenience.

The relation $\sqsubseteq$ is a partial order. Reflexivity and transitivity is true by definition of $\sqsubseteq$ as the reflexive and transitive closure of $\sqsubseteq$. For antisymmetry, we have that:

- for $n, m \in \mathbb{N}$, $n \sqsubseteq m$ iff $n \leq m$;

- ▷ for any $n \in \mathbb{N}$ and $m \in L \setminus \mathbb{N}$, we have $n \sqsubseteq m$ and $m \not\sqsubseteq n$;
- ▷ $\alpha \sqsubseteq \top$, $\beta \sqsubseteq \top$, $\top \not\sqsubseteq \alpha$, $\top \not\sqsubseteq \beta$, $\alpha \not\sqsubseteq \beta$ and $\beta \not\sqsubseteq \alpha$;

(this can be shown by induction on the relation $\sqsubseteq$).

We have that 0 is the least element in $(L, \sqsubseteq)$: we have that $0 \sqsubseteq a$ for all $a \in L$. For $a, b \in L$, we can define $a \vee b$ as:

- ▷ if $a, b \in \mathbb{N}$, let $a \vee b := \min_{\leq_\mathbb{N}}(a, b)$;
- ▷ if $a \in \mathbb{N}$ and $b \in L \setminus \mathbb{N}$, let $a \vee b, b \vee a := b$;
- ▷ otherwise let $\alpha \vee \beta := \top$, $a \wedge a := a$, $a \wedge \top, \top \wedge a := \top$ for $a \in \{\alpha, \beta, \top\}$.

Using the previous results on $\sqsubseteq$, we have that $- \vee -$ *really* is a join.

This concludes the proof that $(L, \sqsubseteq)$ is a join semilattice.

We also have that $(L, \sqsubseteq)$ is not a lattice. Suppose it is a lattice, and consider the element $a := \alpha \wedge \beta$. Necessarily, we have that $a \in \mathbb{N}$ (if $a = \alpha$ then we would have $\alpha \sqsubseteq \beta$, which is false). As $a = \alpha \wedge \beta$ and $a + 1 \sqsubseteq \alpha, \beta$ we have, by definition of meet, that $a + 1 \sqsubseteq a$, thus $a + 1 \leq a$ (since $a, a + 1 \in \mathbb{N}$) which is ***absurd***. We can conclude that $(L, \sqsubseteq)$ is not a lattice.

Question 9. *Consider a set L equipped with two binary operations $\wedge, \vee : L \times L \to L$ and two constants $\top, \bot \in L$. Assume that $(L, \wedge, \top)$ and $(L, \vee, \bot)$ are commutative monoids in which every element is idempotent. Show that the following are equivalent.*

1. *The partial order $\leq_\vee$ induced by $(L, \vee, \bot)$ coincides with the partial order $\leq_\wedge$ induces by $(L, \wedge, \top)$.*
2. *$(L, \vee, \wedge, \bot, \top)$ satisfies the two following **absorptive laws**:*

$$\forall a, b \in L, \quad a \vee (a \wedge b) = a \tag{abs$_1$}$$
$$\forall a, b \in L, \quad a \wedge (a \vee b) = a \tag{abs$_2$}$$

- ▷ Let us show that 1 implies 2. Let $a, b \in L$. We have that $a \wedge b \leq_\wedge a$ and, assuming $\leq_\wedge$ and $\leq_\vee$ coincide, $a \wedge b \leq_\vee a$, thus $a \vee (a \wedge b) = a$, *i.e.* (abs$_2$) holds. Similarly, $a \leq_\vee a \vee b$ thus $a \leq_\wedge a \vee b$, so $(a \wedge b) \vee a = a$ holds, and we can recover (abs$_1$) by using commutativity.
- ▷ Let us show that 2 implies 1.

- Suppose $b \leq_\wedge a$, then $b \wedge a = b$. By (abs$_1$) and commutativity, we have $b \vee a = (b \wedge a) \vee a = a$, thus $b \leq_\vee a$.
- Suppose $b \leq_\vee a$, then $b \vee a = a$. By (abs$_2$), we have

$$b \wedge a = b \wedge (b \vee a) = b,$$

  thus $b \leq_\wedge a$.

Thus the two order coincide.

Question 10. *Show that the partial order $(\mathfrak{L}(\mathsf{LML}), \leq)$ is a lattice.*

We have shown that $(\mathfrak{L}(\mathsf{LML}), \leq)$ has a greatest element $\top$, a least element $\bot$, binary meets given by $- \wedge -$ and binary joins given by $- \vee -$ (question 6). Thus it has all finite meets and finite joins (as seen in question 3), *i.e.* $(\mathfrak{L}(\mathsf{LML}), \leq)$ is a lattice.

Question 11. *Show that the function*

$$\begin{aligned} \bigcirc : \mathfrak{L}(\mathsf{LML}) &\longrightarrow \mathfrak{L}(\mathsf{LML}) \\ \phi &\longmapsto \bigcirc\phi \end{aligned}$$

is a morphism of lattices.

We know $\bigcirc$ is a map of meet iff $\bigcirc\top = \top$ and $\bigcirc(\phi \wedge \psi) = \bigcirc\phi \wedge \bigcirc\psi$. Both are true as,

$$\begin{aligned} [\![\bigcirc\top]\!] &= \{\sigma \in (\mathbf{2}^{\mathrm{AP}})^\omega \mid \sigma \upharpoonright 1 \in [\![\top]\!] = (\mathbf{2}^{\mathrm{AP}})^\omega\} = (\mathbf{2}^{\mathrm{AP}})^\omega = [\![\top]\!] \\ [\![\bigcirc(\phi \wedge \psi)]\!] &= \{\sigma \in (\mathbf{2}^{\mathrm{AP}})^\omega \mid \sigma \upharpoonright 1 \in [\![\phi]\!] \cap [\![\psi]\!]\} = [\![\bigcirc\phi]\!] \cap [\![\bigcirc\psi]\!] = [\![\bigcirc\phi \wedge \bigcirc\psi]\!]. \end{aligned}$$

Very similarly, $\bigcirc$ is a map of joins iff $\bigcirc\bot = \bot$ and $\bigcirc(\phi \vee \psi) = \bigcirc\phi \vee \bigcirc\psi$. One can show that both equalities hold by applying $[\![-]\!]$ and showing the equality of the sets like above.

Thus $\bigcirc : (\mathfrak{L}(\mathsf{LML}), \leq) \to (\mathfrak{L}(\mathsf{LML}), \leq)$ is a morphism of lattices.

2.3 Distributive Lattices.

Question 12. *Show that the following two **distributive laws** are equivalent in a lattice* $(L, \vee, \wedge, \perp, \top)$*:*

$$\forall a, b, c \in L, \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \tag{dist$_1$}$$
$$\forall a, b, c \in L, \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \tag{dist$_2$}$$

Suppose (dist$_1$) holds and let us show (dist$_2$) is true for $a, b, c \in L$:

$$\begin{aligned} (a \vee b) \wedge (a \vee c) &= ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c) && \text{by (dist}_1\text{)} \\ &= a \vee ((a \vee b) \wedge c) && \text{by (abs}_2\text{)} \\ &= a \vee (a \wedge b) \vee (b \wedge c) && \text{by (dist}_1\text{)} \\ &= a \vee (b \wedge c) && \text{by (abs}_1\text{).} \end{aligned}$$

To prove that (dist$_1$) holds when (dist$_2$) is true, we can apply the previous result to the lattice $(L, \leq)^{\mathrm{op}} = (L, \geq)$. This gives exactly the implication "(dist$_2$) implies (dist$_1$)," as wanted.

Thus, the two distributive laws (dist$_1$) and (dist$_2$) are equivalent.

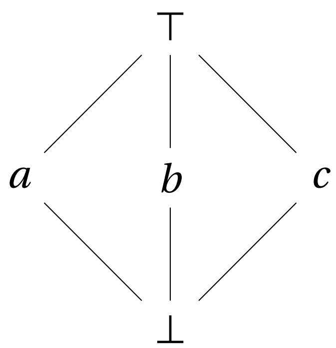
Question 13. *Show that the lattice* $(\mathfrak{L}(\mathsf{LML}), \leq)$ *is distributive.*

Let $\phi, \psi, \vartheta \in \mathfrak{L}(\mathsf{LML})$. We have that

$$\llbracket \phi \wedge (\psi \vee \vartheta) \rrbracket = \llbracket \phi \rrbracket \cap (\llbracket \psi \rrbracket \cup \llbracket \vartheta \rrbracket) = (\llbracket \phi \rrbracket \cap \llbracket \psi \rrbracket) \cup (\llbracket \phi \rrbracket \cap \llbracket \vartheta \rrbracket) = \llbracket (\phi \wedge \psi) \vee (\phi \wedge \vartheta) \rrbracket,$$

thus $\phi \wedge (\psi \vee \vartheta) = (\phi \wedge \psi) \vee (\phi \wedge \vartheta)$.

Question 14. *Consider the following lattice* M_3*:*



(i.e. $\bot \leq a, b, c \leq \top$ with a, b, c incomparable). Show that M_3 is not distributive.

Suppose M_3 is distributive. As a, b, c are incomparable, we have that

$$a \wedge b = a \wedge c = \bot \qquad \text{and} \qquad b \vee c = \top,$$

and thus,

$$a = a \wedge \top = a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) = \bot \vee \bot = \bot,$$

which is ***absurd***. Thus M_3 is not distributive.

2.4 Booleans algebras.

Question 15. *Show that if $(L, \leq)$ is a distributive lattice then $a \in L$ has at most one complement.*

Consider $c, c' \in L$ two complements of $a \in L$. Then, we have that

$$c = c \wedge \top = c \wedge (a \vee c') \overset{(\mathrm{dist}_1)}{=} (c \wedge a) \vee (c \wedge c') = \bot \vee (c \wedge c') = c \wedge c',$$

and,

$$c' = c' \wedge \top = c' \wedge (a \vee c) \overset{(\mathrm{dist}_1)}{=} (c' \wedge a) \vee (c' \wedge c) = \bot \vee (c' \wedge c) = c' \wedge c.$$

We can conclude that $c = c'$ by commutativity of meets.

Question 16. *Show that $(\mathfrak{L}(\mathsf{LML}), \leq)$ is a Boolean algebra.*

Let us show that $\neg\phi$ is a complement for $\phi \in \mathfrak{L}(\mathsf{LML})$. We have to check that $\phi \wedge \neg\phi = \bot$ and $\phi \vee \neg\phi = \top$ hold. Both equalities can be easily checked with interpretations:

$$[\![\phi \wedge \neg\phi]\!] = [\![\phi]\!] \cap [\![\phi]\!]^{\complement} = \emptyset = [\![\bot]\!],$$

and

$$[\![\phi \vee \neg\phi]\!] = [\![\phi]\!] \cup [\![\phi]\!]^{\complement} = (\mathbf{2}^{\mathrm{AP}})^{\omega} = [\![\top]\!].$$

Thus, $\neg\phi$ is ***the*** complement of ϕ in $(\mathfrak{L}(\mathsf{LML}), \leq)$, which is, as a consequence, a Boolean algebra.

Question 17. *Show that the following* ***De Morgan Laws*** *hold in every Boolean algebra* $(B, \vee, \wedge, \bot, \top)$*:*

$$a \wedge b = \neg(\neg a \vee \neg b) \qquad a \vee b = \neg(\neg a \wedge \neg b) \qquad a = \neg\neg a.$$

We start by showing that $\neg a \vee \neg b$ is a complement to $(a \wedge b)$:

$$(a \wedge b) \wedge (\neg a \vee \neg b) = (a \wedge b \wedge \neg b) \vee (a \wedge b \wedge \neg a) = \bot \vee \bot = \bot$$
$$(a \wedge b) \vee (\neg a \vee \neg b) = (a \vee \neg a \vee \neg b) \wedge (b \vee \neg a \vee \neg b) = \top \wedge \top = \top,$$

by using the Boolean algebra laws. Thus $a \wedge b = \neg(\neg a \vee \neg b)$.

For $a \vee b = \neg(\neg a \wedge \neg b)$, we proceed by duality by applying the previous result to the Boolean algebra $(B, \wedge, \vee, \top, \bot)$, as a complement for a in $(B, \geq)$ is exactly a complement for a in $(B, \leq)$.

We can easily check that $\neg\top = \bot$, and then

$$a = a \wedge \top = \neg(\neg a \vee \neg\top) = \neg(\neg a \vee \bot) = \neg\neg a.$$

Question 18. *Show that if f is a map of Boolean algebras from $(B, \leq)$ to $(B', \leq')$ then f preserves complements.*

We have that

$$\bot' = f(\bot) = f(a \wedge \neg a) = f(a) \wedge' f(\neg a),$$

and

$$\top = f(\top) = f(a \vee \neg a) = f(a) \vee' f(\neg a),$$

thus $f(\neg a)$ is a complement of $f(a)$ and, by unicity, $f(\neg a) = \neg f(a)$. We can conclude that a map of Boolean algebras preserves complements.

3 Representation of Boolean Algebras.

3.1 Filters and Ultrafilters.

Question 19. *Let $(L, \wedge, \top)$ be a meet semilattice. Show that $\mathscr{F} \subseteq L$ is a filter iff*

1. *$\mathcal{F}$ is upward-closed and,*

2. *$\top \in \mathcal{F}$ and,*

3. *$a \wedge b \in \mathcal{F}$ whenever $a \in \mathcal{F}$ and $b \in \mathcal{F}$.*

Let $\mathcal{F} \subseteq L$ be upward-closed.

We only need to show that $\mathcal{F}$ is codirected if, and only if, $\top \in \mathcal{F}$ and $a \wedge b \in \mathcal{F}$ whenever a and b are in $\mathcal{F}$.

- ▷ If $\mathcal{F}$ is codirected then there exists $a \in \mathcal{F}$ and, as $a \leq \top$, we have that $\top \in \mathcal{F}$. Also, for $a, b \in \mathcal{F}$, there exists $c \in \mathcal{F}$ such that $c \leq a$ and $c \leq b$ therefore, as $c \leq a \wedge b$, we have that $a \wedge b \in \mathcal{F}$.
- ▷ Suppose $\top \in \mathcal{F}$ and that $\mathcal{F}$ is stable under the meet operator. Let us show that $\mathcal{F}$ is codirected. First, $\top \in \mathcal{F}$ so $\mathcal{F}$ is non-empty. Second, for any two $a, b \in \mathcal{F}$ then $c := a \wedge b \in \mathcal{F}$ satisfies that $c \leq a, b$.

Question 20. *Let $(L, \leq)$ be a lattice. Show that if $F \subseteq L$ has the finite intersection property, then*

$$\mathrm{Filt}(F) := \left\{ a \in L \mid a \geq \bigwedge S \text{ for some finite } S \subseteq F \right\}$$

is a proper filter on $(L, \leq)$.

Suppose $F \subseteq L$ has the finite intersection property.

- ▷ First, $\top \in \mathrm{Filt}(F)$ as $\top \geq \bigwedge S$ for, in particular, $S = \emptyset \subseteq F$.
- ▷ Second, if $a, b \in \mathrm{Filt}(F)$ with $a \geq \bigwedge S_1$, $b \geq \bigwedge S_2$ and finite $S_1, S_2 \subseteq F$, then $a \wedge b \geq (\bigwedge S_1) \wedge (\bigwedge S_2) = \bigwedge (S_1 \cup S_2)$, thus $a \wedge b \in \mathrm{Filt}(F)$ (as $S_1 \cup S_2$ is a finite subset of F).
- ▷ Third, if $a \in \mathrm{Filt}(F)$ (with $a \geq \bigwedge S$ and a finite $S \subseteq F$) and $a \leq b$, then $b \geq \bigwedge S$ thus $b \in \mathrm{Filt}(F)$.

We have that $\mathrm{Filt}(F)$ is a filter.

Suppose $\bot \in \mathrm{Filt}(F)$, then there exists some finite $S \subseteq F$ such that $\bot \geq \bigwedge S$. Also, $\bot \leq \bigwedge S$, so $\bigwedge S = \bot$. However, this is absurd by the finite intersection property for F. Thus $\bot \notin \mathrm{Filt}(F)$ and we can conclude that $\mathrm{Filt}(F)$ is a proper filter.

Question* 21. *Let $\mathscr{F}$ be a filter on a distributive lattice. Show that if $\mathscr{F}$ is an ultrafilter then $\mathscr{F}$ is prime.*

Lemma 2. Any proper filter has the finite intersection property.

Proof. Consider some finite subset $S \subseteq \mathscr{F}$ where $\mathscr{F}$ is a proper filter. Then, by induction on the size of S, we can show that $\bigwedge S \in \mathscr{F}$, and thus $\bigwedge S \neq \bot$, as $\bot \notin \mathscr{F}$. □

As $\mathscr{F}$ is an ultrafilter, $\mathscr{F}$ is a proper filter and thus $\bot \notin \mathscr{F}$. Now, let us show that if $a \vee b$ is in $\mathscr{F}$ then either a or b is in $\mathscr{F}$. Consider three cases.

- ▷ Either $\mathscr{F} \cup \{a\}$ has the finite intersection property, and thus $\mathrm{Filt}(\mathscr{F} \cup \{a\})$ is a proper filter and $\mathscr{F} \subseteq \mathrm{Filt}(\mathscr{F} \cup \{a\})$ (simply take $S = \{f\}$ for every $f \in \mathscr{F}$ in the definition of $\mathrm{Filt}(-)$), thus $a \in \mathscr{F} = \mathrm{Filt}(\mathscr{F} \cup \{a\})$ since $\mathscr{F}$ is an ultrafilter.
- ▷ Either $\mathscr{F} \cup \{b\}$ has the finite intersection property, and we can show $b \in \mathscr{F}$ very similarly.
- ▷ Either $\mathscr{F} \cup \{a\}$ and $\mathscr{F} \cup \{b\}$ do not have the finite intersection property, *i.e.* there exists some finite $S \subseteq \mathscr{F} \cup \{a\}$ and $S' \subseteq \mathscr{F} \cup \{b\}$ such that $\bigwedge S = \bigwedge S' = \bot$. Necessarily $a \in S$ and $b \in S'$ (if $S \subseteq \mathscr{F}$ then with the above lemma, we immediately have that $\bigwedge S \neq \bot$, and similarly for S'). Then, writing $U := \bigwedge(S \smallsetminus \{a\})$ and $V := \bigwedge(S' \smallsetminus \{b\})$, we have

$$
\begin{aligned}
&(a \vee b) \wedge (U \wedge V) \\
&= (a \wedge (U \wedge V)) \vee (b \wedge (U \wedge V)) && \text{by distributivity, commutativity} \\
&= (V \wedge \bot) \vee (U \wedge \bot) && \text{as } a \wedge U = \wedge S = \bot \\
&= \bot \vee \bot \\
&= \bot.
\end{aligned}
$$

  However, $a \vee b$, U and V are in $\mathscr{F}$ thus so is $(a \vee b) \wedge U \wedge V = \bot \in \mathscr{F}$. This is absurd as $\mathscr{F}$ is a proper filter.

Question 22. *Let $(B, \leq)$ be a Boolean algebra and let $\mathscr{F} \subseteq B$ be a filter. Show that the following are equivalent:*

1. *$\mathscr{F}$ is an ultrafilter;*
2. *$\mathscr{F}$ is prime;*

3. *for each $a \in B$, we have $a \in \mathscr{F}$ iff $\neg a \notin \mathscr{F}$.*

By question 21, we have that 1 implies 2.

To prove that 2 implies 3, consider some $a \in B$: we have $a \vee \neg a = \top \in \mathscr{F}$, so either $a \in \mathscr{F}$ or $\neg a \in \mathscr{F}$ $(\star)$. If $a \in \mathscr{F}$ then $\neg a \notin \mathscr{F}$ since, if $\neg a \in \mathscr{F}$ then $a \wedge \neg a = \bot \in \mathscr{F}$, a contradiction since $\mathscr{F}$ is supposed prime. On the other hand, if $\neg a \notin \mathscr{F}$ then necessarily $a \in \mathscr{F}$ by $(\star)$.

To prove 3 implies 1, consider some proper filter $\mathscr{H} \supsetneq \mathscr{F}$, then there exists some $a \in \mathscr{H} \setminus \mathscr{F}$, and so $\neg a \in \mathscr{F} \subseteq \mathscr{H}$ (applying 3 to $\neg a$ with $\neg\neg a = a$). Thus $a, \neg a \in \mathscr{H}$ and so $a \wedge \neg a = \bot \in \mathscr{H}$, a contradiction since $\mathscr{H}$ is a proper filter.

Question 23. *Let $(A, \leq)$ be a partial order and consider some $\mathscr{F} \subseteq A$.*

1. *Show that $\mathscr{F}$ is upward-closed if and only if $\chi_{\mathscr{F}}$ is monotone.*
2. *Assume that A is a meet semilattice. Show that $\mathscr{F}$ is a filter if and only if $\chi_{\mathscr{F}} : A \to \mathbf{2}$ is a morphism of meet semilattices.*
3. *Assume that A is a lattice. Show that $\mathscr{F}$ is a prime filter if and only if $\chi_{\mathscr{F}} : A \to \mathbf{2}$ is a morphism of lattices.*

In the following subquestions, we will implicitly use the results when from the previous subquestions.

1. Suppose $\chi_{\mathscr{F}}$ monotone. Let us show that $\mathscr{F}$ is upward-closed. Consider $a \leq b$ with $a \in \mathscr{F}$, then $1 = \chi_{\mathscr{F}}(a) \leq \chi_{\mathscr{F}}(b)$, thus $\chi_{\mathscr{F}}(b) = 1$ and so $b \in \mathscr{F}$.

   Conversely suppose $\mathscr{F}$ is upward-closed, let us show $\chi_{\mathscr{F}}$ is monotone. Consider $a \leq b$.

   ▷ If $a \notin \mathscr{F}$ then $0 = \chi_{\mathscr{F}}(a) \leq \chi_{\mathscr{F}}(b)$ is true.

   ▷ If $a \in \mathscr{F}$ then $b \in \mathscr{F}$ and $1 = \chi_{\mathscr{F}}(a) \leq \chi_{\mathscr{F}}(b) = 1$.

2. Suppose $\chi_{\mathscr{F}}$ is a morphism of meet semilattices. Let us show that $\mathscr{F}$ is a filter. Take $a, b \in \mathscr{F}$, then $\chi_{\mathscr{F}}(a \wedge b) = \chi_{\mathscr{F}}(a) \wedge \chi_{\mathscr{F}}(b) = 1 \wedge 1 = 1$ thus $a \wedge b \in \mathscr{F}$. Also, $\chi_{\mathscr{F}}(\top) = \top_{\mathbf{2}} = 1$, thus $\top \in \mathscr{F}$.

Conversely suppose $\mathscr{F}$ is a filter, then $\chi_{\mathscr{F}}(\top) = \top_{\mathbf{2}} = 1$ as $\top \in \mathscr{F}$. And, for $a, b \in L$,

- ▷ if $a, b \in \mathscr{F}$ then $1 = \chi_{\mathscr{F}}(a \wedge b) = \chi_{\mathscr{F}}(a) \wedge \chi_{\mathscr{F}}(b) = 1 \wedge 1$;
- ▷ if $a \notin \mathscr{F}$ then $a \wedge b \notin \mathscr{F}$ (if $a \wedge b \in \mathscr{F}$, then $a \wedge b \leq a$ would imply that $a \in \mathscr{F}$), thus $0 = \chi_{\mathscr{F}}(a \wedge b) = \chi_{\mathscr{F}}(a) \wedge \chi_{\mathscr{F}}(b) = 0 \wedge \chi_{\mathscr{F}}(b)$;
- ▷ similarly if $b \notin \mathscr{F}$.

3. Suppose $\chi_{\mathscr{F}}$ is a morphism of lattices. Let us show that $\mathscr{F}$ is a proper filter. As $\chi_{\mathscr{F}}(\bot) = \bot_{\mathbf{2}} = 0$, then $\bot \notin \mathscr{F}$. If $a \vee b \in \mathscr{F}$, then

$$1 = \chi_{\mathscr{F}}(a \vee b) = \chi_{\mathscr{F}}(a) \vee \chi_{\mathscr{F}}(b),$$

thus necessarily $\chi_{\mathscr{F}}(a) = 1$ or $\chi_{\mathscr{F}}(b) = 1$ (if $\chi_{\mathscr{F}}(a) = \chi_{\mathscr{F}}(b) = 0$ then $\chi_{\mathscr{F}}(a \vee b) = 0$ a contradiction), and so either $a \in \mathscr{F}$ or $b \in \mathscr{F}$.

Conversely suppose $\mathscr{F}$ is a proper filter. Then $\chi_{\mathscr{F}}(\bot) = 0 = \bot_{\mathbf{2}}$ as $\bot \notin \mathscr{F}$. Take $a, b \in L$:

- ▷ if $a, b \notin \mathscr{F}$ then $0 = \chi_{\mathscr{F}}(a \vee b) = \chi_{\mathscr{F}}(a) \vee \chi_{\mathscr{F}}(b) = 0 \vee 0$ (if $a \vee b \in \mathscr{F}$ then either $a \in \mathscr{F}$ or $b \in \mathscr{F}$);
- ▷ if $a \in \mathscr{F}$ then $a \leq a \vee b$ implies $a \vee b \in \mathscr{F}$ and so $1 = \chi_{\mathscr{F}}(a \vee b) = \chi_{\mathscr{F}}(a) \vee \chi_{\mathscr{F}}(b) = 1 \vee \chi_{\mathscr{F}}(b)$;
- ▷ similarly if $b \in \mathscr{F}$.

In these subquestions, we also proved the following result:

Lemma 3. For $a, b \in L$, and $\mathscr{F} \subseteq L$,

- ▷ if $\mathscr{F}$ is a filter, then $a \wedge b \in \mathscr{F}$ iff $a, b \in \mathscr{F}$;
- ▷ if $\mathscr{F}$ is a prime filter, then $a \vee b \notin \mathscr{F}$ iff $a, b \notin \mathscr{F}$. □

3.2 The Spectrum of a Boolean Algebra.

Question 24. *Let* $(B, \leq)$ *be a Boolean algebra. Show that we have:*

$$\begin{aligned} \mathrm{ext}(a \wedge b) &= \mathrm{ext}(a) \cap \mathrm{ext}(b) \\ \mathrm{ext}(a \vee b) &= \mathrm{ext}(a) \cup \mathrm{ext}(b) \\ \mathrm{ext}(\neg a) &= \mathbf{Sp}(B) \smallsetminus \mathrm{ext}(a) \\ \mathrm{ext}(\top) &= \mathbf{Sp}(B) \\ \mathrm{ext}(\bot) &= \emptyset. \end{aligned}$$

Let $\mathscr{F} \in \mathbf{Sp}(B)$. We have that :

$$\begin{aligned} \mathscr{F} \in \mathrm{ext}(a \wedge b) &\iff a \wedge b \in \mathscr{F} \iff a, b \in \mathscr{F} \iff \mathscr{F} \in \mathrm{ext}(a) \cap \mathrm{ext}(b); \\ \mathscr{F} \notin \mathrm{ext}(a \vee b) &\iff a \vee b \notin \mathscr{F} \iff a, b \notin \mathscr{F} \iff \mathscr{F} \notin \mathrm{ext}(a) \cup \mathrm{ext}(b). \end{aligned}$$

If $\mathscr{F} \notin \mathrm{ext}(\top)$ then $\top \notin \mathscr{F}$ which is absurd. If $\mathscr{F} \in \mathrm{ext}(\bot)$ then $\bot \in \mathscr{F}$ which is impossible since $\mathscr{F}$ is a proper filter. A map $\mathrm{ext} : (B, \leq) \to (\wp(\mathbf{Sp}(B)), \subseteq)$ of Boolean algebras automatically preserves complements, thus $\mathrm{ext}(\neg a) = \mathbf{Sp}(B) \smallsetminus \mathrm{ext}(a)$.

Question 25. *Show that the spectrum* $(\mathbf{Sp}(B), \Omega(\mathbf{Sp}(B)))$ *of a Boolean algebra* B *is* T_0 *and zero-dimensional.*

Let us show that $\mathbf{Sp}(B)$ is T_0. Take $F \neq G \in \mathbf{Sp}(B)$, and $a \in F \triangle G$ where $- \triangle -$ is the symmetric difference of two sets. Without loss of generality, assume $a \in F$ (and thus $a \notin G$), then $F \in \mathrm{ext}(a)$ and $G \notin \mathrm{ext}(a)$.

Let us show that $\mathbf{Sp}(B)$ is zero-dimensional. Every $\mathrm{ext}(a) \in \mathscr{B}$ is a clopen, as it is obviously open, and $\mathrm{ext}(a) = \mathrm{ext}(\neg\neg a) = \mathbf{Sp}(B) \smallsetminus \mathrm{ext}(\neg a)$, thus it is also closed. So, the basis $\mathscr{B}$ only contains clopens.

Question* 26. *Show that the spectrum* $(\mathbf{Sp}(B), \Omega(\mathbf{Sp}(B)))$ *of a Boolean algebra* B *is compact.*

> **Note.** In the following questions, we will only deal with open covers, so a “cover” will always be an open cover. Similarly a “subcover” will always refer to an open subcover.

Suppose $\mathbf{Sp}(B) = \bigcup_{a \in A} \mathrm{ext}(a)$ (we can always do that as $\mathscr{B}$ is a basis of the topology on $\mathbf{Sp}(B)$). Assume this cover of $\mathbf{Sp}(B)$ does not admit a finite subcover. Define

$$\mathscr{C} := \{\neg a \mid a \in A\}.$$

The set $\mathscr{C}$ has the finite intersection property: if S is a finite subset of $\mathscr{C}$ with $\bigwedge S = \bot$, then

$$\bot = \bigwedge_{(\neg a) \in S} (\neg a) \overset{(\star)}{=} \neg\Big(\bigvee_{(\neg a) \in S} a \Big),$$

where $(\star)$ is proven by induction with results from question 17, as S is finite. Taking the complement, we have $\top = \bigvee_{(\neg a) \in S} a$, thus

$$\mathbf{Sp}(B) = \mathrm{ext}(\top) = \mathrm{ext}\Big(\bigvee_{(\neg a) \in S} a \Big) = \bigcup_{(\neg a) \in S} \mathrm{ext}(a)$$

is a finite subcover of $\bigcup_{a \in A} \mathrm{ext}(a)$, which is absurd. Thus $\mathscr{C}$ has the finite intersection property. With the Ultrafilter Lemma, we get an ultrafilter $\mathscr{F} \supseteq \mathscr{C}$. For all $a \in A$, we have $\neg a \in \mathscr{C} \subseteq \mathscr{F}$ thus $a \notin \mathscr{F}$ (question 22), and so $\mathscr{F} \notin \mathrm{ext}(a)$. So, we have $\mathscr{F} \notin \mathbf{Sp}(B) = \bigcup_{a \in A} \mathrm{ext}(a)$, a contradiction. Thus, the cover $\mathbf{Sp}(B) = \bigcup_{a \in A} \mathrm{ext}(a)$ admits a finite subcover. We can conclude that $\mathbf{Sp}(B)$ is compact.

Question 27. *Let (X, Ω) be a topological space. Show that (X, Ω) is compact if and only if we have $\bigcap \mathscr{F} \neq \emptyset$ for every family of closed sets $\mathscr{F}$ which has the finite intersection property (w.r.t. the inclusion (partial) order on closed sets).*

For any family $\mathscr{G} \subseteq \wp(X)$, we will write $\bar{\mathscr{G}} := \{X \setminus G \mid G \in \mathscr{G}\}$. We have that $\mathscr{G}$ is a family of closed sets iff $\bar{\mathscr{G}}$ is a family of open sets (*i.e.* $\bar{\mathscr{G}} \subseteq \Omega$). Also, we have that $\bar{\bar{\mathscr{G}}} = \mathscr{G}$.

Firstly, suppose (X, Ω) to be compact, and let $\mathscr{F} \subseteq \wp(X)$ be a family of closed sets with the finite intersection property. Suppose $\bigcap \mathscr{F} = \emptyset$. Then, $\bigcup \bar{\mathscr{F}} = X$ and, as X is compact, there exists some finite $\mathscr{G} \subseteq \bar{\mathscr{F}}$ such that $\bigcup \mathscr{G} = X$. Thus we get that $\bigcap \bar{\mathscr{G}} = \emptyset$, a contradiction with the finite intersection property, as $\bar{\mathscr{G}}$ is finite. We conclude that $\bigcap \mathscr{F} \neq \emptyset$.

Secondly, suppose $X = \bigcup \mathscr{F}$ with $\mathscr{F} \subseteq \Omega$ is an open cover of X. Assume that there are no finite subcovers $(\star)$. Then, by complement, we have $\bigcap \bar{\mathscr{F}} = \emptyset$

$(\star\star)$. But, $\bar{\mathscr{F}}$ is a family of closed sets and $\bar{\mathscr{F}}$ has the finite intersection property: if there exists some finite $\mathscr{G} \subseteq \bar{\mathscr{F}}$ with $\bigcap \mathscr{G} = \emptyset$ then $\bigcup \bar{\mathscr{G}} = X$ is a finite cover of X with open sets, which is absurd by $(\star)$. So we conclude that $\bar{\mathscr{F}}$ has the finite intersection property, and we can apply the hypothesis to get that $\bigcap \bar{\mathscr{F}} \neq \emptyset$, a contradiction with $(\star\star)$. Thus a finite subcover exists, and we finally have that X is compact.

Question* 28. *Given a Stone space (X, Ω), consider the function*

$$\begin{aligned} \eta : X &\longrightarrow \wp(\mathbf{K}\Omega) \\ x &\longmapsto \{U \in \mathbf{K}\Omega \mid x \in U\}. \end{aligned}$$

Show that η is a continuous bijection from X to $\mathbf{Sp}(\mathbf{K}\Omega)$.

We will proceed in four parts.

Firstly, let us show that $\eta(x) \in \mathbf{Sp}(\mathbf{K}\Omega)$, *i.e.* $\eta(x)$ is a prime filter on $\mathbf{K}\Omega$.

- ▷ Suppose $A \in \eta(x)$ and $A \subseteq B \in \mathbf{K}\Omega$, then $x \in A \subseteq B$, we have $x \in B$ and so $B \in \eta(X)$.
- ▷ We have that $X \in \eta(x)$ as $x \in X$ and X is a clopen.
- ▷ Suppose $A, B \in \eta(x)$ then $x \in A$ and $x \in B$, so $x \in A \cap B$ and thus $A \cap B \in \eta(x)$.

Therefore $\eta(x)$ is a filter on $\mathbf{K}\Omega$.

- ▷ Suppose $\bot_{\mathbf{K}\Omega} = \emptyset \in \eta(x)$, then $x \in \emptyset$, which is absurd, so $\eta(x)$ is a proper filter.
- ▷ Suppose $A \cup B \in \eta(x)$, then $x \in A \cup B$ so either $x \in A$ or $x \in B$, and thus either $A \in \eta(x)$ or $B \in \eta(x)$.

We can conclude that $\eta : X \to \mathbf{Sp}(\mathbf{K}\Omega)$.

Secondly, let us show that η is injective. Take $x \neq y$ in X. By T_0, we have that there exists an open set $U \in \Omega$ with $x \in U$ and $y \notin U$, even if it requires swapping x and y. As (X, Ω) is zero-dimensional, we can write $U = \bigcup_{i \in I} C_i$ where $C_i \in \mathbf{K}\Omega$. Let $C \in \{C_i \mid i \in I\} \subseteq \mathbf{K}\Omega$ such that $x \in C_i$. We also have that $y \notin C$, thus $C \in \eta(x)$ and $C \notin \eta(y)$, so $\eta(x) \neq \eta(y)$.

Thirdly let us show that η is surjective. Take $\mathscr{F} \in \mathbf{Sp}(\mathbf{K}\Omega)$. Then, from question 27 (as X is, by definition, compact and $\mathscr{F}$ is a set of closed sets, with the finite intersection property), we have that $\bigcap \mathscr{F} \neq \emptyset$. Take $x \in \bigcap \mathscr{F} \neq \emptyset$. For every $U \in \mathscr{F}$, $x \in U$ so $U \in \eta(x)$ and thus $\mathscr{F} \subseteq \eta(x)$. Conversely suppose $V \in \mathbf{K}\Omega$ such that $x \in V$ and $V \notin \mathscr{F}$, then $X \setminus V \in \mathscr{F}$ by question 22(3), and so $x \in \bigcap \mathscr{F} \subseteq X \setminus V$ and $x \notin X \setminus V$, a contradiction. We can thus conclude $\mathscr{F} = \eta(x)$, and so η is surjective.

Finally, let us show that η is continuous. To show that $\eta^\bullet(V)$ is open for every $V \in \Omega(\mathbf{Sp}(\mathbf{K}\Omega))$, it suffices to consider $V \in \mathscr{B}$ as $\eta^\bullet$ commutes with arbitrary unions. Let $A \in \mathbf{K}\Omega$, and let us show that $\eta^\bullet(\mathrm{ext}(A))$ is open in X:

$$\eta^\bullet(\mathrm{ext}(A)) = \{x \in X \mid \eta(x) \in \mathrm{ext}(A)\} = \{x \in X \mid \eta(x) \ni A\} = \{x \in X \mid x \in A\},$$

which exactly is $A \in \mathbf{K}\Omega \subseteq \Omega$.

Thus we can conclude that η is a continuous bijection from X to $\mathbf{Sp}(\mathbf{K}\Omega)$.

Question★ 29. *Assume $(X, \Omega X)$ and $(Y, \Omega Y)$ are compact Hausdorff spaces. Show that if $f : (X, \Omega X) \to (Y, \Omega Y)$ is a continuous bijection, then f is an homeomorphism.*

We will us the following lemma from the course:

Lemma 4. If $(X, \Omega X)$ is a compact Hausdorff space, and $C \subseteq X$, then C is compact iff C is closed. □

We also need the following lemma:

Lemma 5. Let $(X, \Omega X)$ and $(Y, \Omega Y)$ be arbitrary topological spaces. If K is compact in $(X, \Omega X)$ then, for every continuous $f : (X, \Omega X) \to (Y, \Omega Y)$, the set $f_!(K)$ is compact in $(Y, \Omega Y)$.

Proof. Define $L := f_!(K)$. Consider a cover $L \subseteq \bigcup_{i \in I} V_i$ of L. Then, $f^\bullet(L) \subseteq f^\bullet(\bigcup_{i \in I} V_i) = \bigcup_{i \in I} f^\bullet(V_i)$, so we obtain an open (as f is continuous) cover of $f^\bullet(L)$. As $K \subseteq f^\bullet(K) \subseteq \bigcup_{i \in I} f^\bullet(V_i)$ and K is compact, then there exists a finite set $J \subseteq I$ such that $K \subseteq \bigcup_{i \in J} f^\bullet(V_i)$. With the direct image $f_!$, we have that $f_!(K) = L \subseteq \bigcup_{i \in J} f_!(f^\bullet(V_i)) = \bigcup_{i \in J} V_i$ is a finite subcover of $L = f_!(K)$. □

We simply have to show that $g := f^{-1} : Y \to X$ is continuous. We have that $g_! = f^\bullet$ and $f_! = g^\bullet$ as f and g are inverses of each other. Consider an arbitrary open set $U \in \Omega X$. It suffices to show that $g^\bullet(U) = f_!(U)$ is open in $(Y, \Omega Y)$. With the two previous lemmas, we have that $f_!(C)$ is a closed set in $(Y, \Omega Y)$, for every closed C in $(X, \Omega X)$. So,

$$f_!(X \setminus U) = g^\bullet(X \setminus U) = g^\bullet(X) \setminus g^\bullet(U) = Y \setminus f_!(U)$$

is closed in $(Y, \Omega Y)$, thus $f_!(U)$ is open. Thus g is continuous.

We can conclude that f is a homeomorphism.

3.3 On the Ultrafilter Lemma.

Question★ 30. *Prove the Ultrafilter Lemma (assuming Zorn's Lemma).*

Consider some $F \subseteq L$ with the finite intersection property where $(L, \leq)$ is a lattice. Define $P := \{\mathscr{F} \text{ is a proper filter on } L \mid F \subseteq \mathscr{F}\}$, ordered by set inclusion $\subseteq$.

Consider a non-empty chain $\mathscr{C}$ in P. We will show that $\mathscr{G} := \bigcup \mathscr{C}$ is a proper filter containing F.

- ▷ Take $a \leq b$ with $a \in \mathscr{G}$. Then $a \in \mathscr{F}$ for some $\mathscr{F} \in \mathscr{C}$. As $\mathscr{F}$ is a filter, then $b \in \mathscr{F}$, and so $b \in \mathscr{G}$.
- ▷ We have $\top \in \mathscr{G}$ as $\mathscr{C}$ is non-empty, and any element of $\mathscr{C}$ contains $\top$.
- ▷ We have $F \subseteq \mathscr{G}$ as $\mathscr{C}$ is non-empty, and any element of $\mathscr{C}$ contains F.
- ▷ Take $a, b \in \mathscr{G}$, then $a \in \mathscr{F}_1$ and $b \in \mathscr{F}_2$ for some proper filters $\mathscr{F}_1, \mathscr{F}_2 \in \mathscr{C}$. As $\mathscr{C}$ is a chain, we can assume without loss of generality, that $\mathscr{F}_1 \subseteq \mathscr{F}_2$, thus $a, b \in \mathscr{F}_2$ and so $a \wedge b \in \mathscr{F}_2 \subseteq \mathscr{G}$.
- ▷ If $\bot \in \mathscr{G}$ then $\bot \in \mathscr{F}$ for some $\mathscr{F}$ in $\mathscr{C}$, which is absurd as $\mathscr{F}$ is a proper filter.

Now, we can immediately see that this is an upper bound of $\mathscr{C}$: if $\mathscr{F} \in \mathscr{C}$ then we have $\mathscr{F} \subseteq \mathscr{G} = \bigcup \mathscr{C}$.

For an empty chain $\mathscr{C}$, we use $\mathrm{Filt}(F) \in P$ as upper bound (as F has the finite intersection property).

Then, we apply Zorn's lemma to $(P, \subseteq)$, and get a maximal element $\mathscr{U} \in P$. Let us show that $\mathscr{U}$ is an ultrafilter. Consider $\mathscr{H}$ a proper filter on L such that $\mathscr{H} \supseteq \mathscr{U}$. Then $F \subseteq \mathscr{U} \subseteq \mathscr{H}$ so $\mathscr{H} \in P$. By maximality of $\mathscr{U}$ in P, we have that $\mathscr{U} = \mathscr{H}$.

We can finally conclude that $\mathscr{U}$ is an ultrafilter containing F, finishing the proof of the Ultrafilter Lemma.

End of Homework.