Algebraic and combinatorial aspects of category theory

Lecture #1. 18/11/25

Algebraic structure $\underbrace{T(X)}_{\text{monad}} \to X$

intuition: $T(X)$ contains the derived operations

Example for a binary operation on X, we can consider

$$\{\text{binary trees with leaves in } X\} \to X.$$

Being created

$$\begin{tikzcd}
T(X) \arrow[d] & T(X \times Y) \arrow[l, "T\pi_1"'] \arrow[r, "T\pi_2"] \arrow[d] & T(Y) \arrow[d] & T\text{-}\mathcal{A}lg \arrow[d, "\text{forgetful functor}"] \\
X & X \times Y \arrow[l] \arrow[r] & Y & \mathcal{S}et
\end{tikzcd}$$

Category: objects, morphisms, composition (unitary and associative).

Notations $A \in \mathcal{C}$ iff A is an object in $\mathcal{C}$

$\mathcal{C}(A,B)$ is the set of morphisms $A \to B$.

Examples

	objects	morphisms	
a set X	$x \in X$	id_x	
a poset $(X, \leq)$	$x \in X$	$x \leq y$	(also works for preordered sets)
$\mathcal{S}et$	set X	maps	
$\mathcal{M}on$	monoid X	homomorphisms	
	$\vdots$		

Given a graph G, then G^* is the category where objects are nodes in G and morphisms $A \to B$ the set of paths from A to B, composition is path concatenation

> **Example:** BM. one object $*$ and $\mathcal{C}(*, *) = M$ with $h \circ g = h \cdot g$.

Example: objects are trees and morphisms are $T_1 \xrightarrow{T} T_2$ when you obtain T_2 by replacing T_1's leaves with copies of T.

<u>A note on size</u>

"small" $\begin{cases} \text{objects form a set} \\ \text{morphisms form a set (from } A \text{ to } B \text{ fixed)} \end{cases}$

"locally small" $\begin{cases} \text{objects form a } \cancel{\text{set}} \text{ class} \\ \text{morphisms form a set (from } A \text{ to } B \text{ fixed)} \end{cases}$

Functor $F : \mathcal{C} \to \mathcal{D}$ that commutes with composition and identity

Notation $F_{A,B} : \mathcal{C}(A, B) \longrightarrow \mathcal{D}(F(A), F(B))$

The axioms of functors can be seen as the fact that the two following diagrams commute.

$$\begin{tikzcd}
\mathcal{C}(B,C) \times \mathcal{C}(A,C) \arrow[r, "- \circ_{\mathcal{C}} -"] \arrow[d, "F_{B,C} \times F_{A,B}"'] & \mathcal{C}(A,C) \arrow[d, "F_{A,C}"] \\
\mathcal{D}(FB, FC) \times \mathcal{D}(FA, FB) \arrow[r, "- \circ_{\mathcal{D}} -"'] & \mathcal{D}(FA, FC)
\end{tikzcd}$$

$\iff Ff \circ Fg = F(f \circ g)$

$F(id_A) = id_{FA} \iff$

$$\begin{tikzcd}
 & \mathcal{C}(A,A) \arrow[d, "F_{A,A}"] \\
1 \arrow[ur, "id_A"] \arrow[r, "id_{F(A)}"'] & \mathcal{D}(FA, FA)
\end{tikzcd}$$

Examples

- A monotone map is a functor between two posets seen as categories
- Forgetful functor $\mathrm{Mon} \to \mathrm{Set}$, $\mathrm{Ring} \to \mathrm{Mon}$
  $(M, \cdot, 1_M) \mapsto M$

Example A functor $F: BM \to BN$ is exactly a monoid homomorphism $f: M \to N$.

Example The "bang" functor $\mathcal{C} \xrightarrow{!} 1$

category with one object and only the id morphism.

Functors compose !!

$\mathcal{C}at$ is the category of small categories. It forms a locally small category

$\mathcal{CAT}$ is the category of locally small categories. It forms a very large category.

$\mathrm{Set}, \mathrm{Grp}, \mathrm{Gph}, \mathcal{C}at \in \mathcal{CAT}$.

A terminal object $T \in \mathcal{C}$ is defined such that for every object $A \in \mathcal{C}$, there is a unique morphism $A \to T$

An initial object $\bot \in \mathcal{C}$ is defined such that for every object $A \in \mathcal{C}$, there is a unique morphism $\bot \to A$.

In Mon, Grp and Field, 1 is both terminal and initial.

In Set, $\emptyset$ is initial ($\emptyset \xrightarrow{\exists!} A$) and 1 is terminal ($A \xrightarrow{\exists!} 1$).

A product of A and $B \in \mathscr{C}$ is an object P with $\overbrace{\begin{array}{l} \pi_1 : P \to A \\ \pi_2 : P \to B \end{array}}^{\text{implicite}}$

such that for every $A \leftarrow Q \to B$, there exists a unique $\langle f, g \rangle : Q \to P$

such that

$$\begin{tikzcd}
 & Q \arrow[dl, bend right, "f"'] \arrow[d, "{\langle f,g \rangle}"] \arrow[dr, bend left, "g"] & \\
A & P \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B
\end{tikzcd}$$

commutes.

In Set, take
$$\langle f, g \rangle : Q \longrightarrow A \times B$$
$$q \longmapsto (f(q), g(q)).$$

> Exercise $B \times A \cong A \times B$ and the unique morphism is
>
> $\langle \pi_2, \pi_1 \rangle : (a, b) \mapsto (b, a)$.

Exercise: If P and Q are both products of A and B then there exists a unique isomorphism from P to Q.

$$\begin{tikzcd}
 & P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, bend left, "\exists! g"] & \\
A & & B \\
 & Q \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \arrow[uu, bend left, "\exists! f"] &
\end{tikzcd}$$

Then

$$\begin{tikzcd}
 & P \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, bend left, "id_P"] & \\
A & & B \\
 & P \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \arrow[uu, bend left, "f \circ g"] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & Q \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, bend left, "id_Q"] & \\
A & & B \\
 & Q \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \arrow[uu, bend left, "g \circ f"] &
\end{tikzcd}$$

imply $f \circ g = id_P$ and $g \circ f = id_Q$.

$$\begin{tikzcd}
& P \arrow[dl] \arrow[dr] & \\
A & Q \arrow[l] \arrow[r] \arrow[u, "g"] & B \\
& P \arrow[ul] \arrow[ur] \arrow[u, "f"] \arrow[uu, bend right=80, "id_P"'] &
\end{tikzcd}$$

// break //

In a poset seen as a category, the (binary) product is the (binary) meet.

Monads

$T(X)$ = terms with free variables in X, up to some equations

Free monoid monad: define the free monoid of $X \in Set$ by induction:

$$\frac{}{X \vdash x}\ (x \in X) \qquad \frac{}{X \vdash 1} \qquad \frac{X \vdash M \qquad X \vdash N}{X \vdash M \cdot N}$$

with the equations

$$\frac{}{(M \cdot N) \cdot P \sim M \cdot (N \cdot P)} \qquad \frac{}{M \sim M \cdot 1} \qquad \frac{}{M \sim 1 \cdot M} \qquad \frac{M \sim M' \qquad N \sim N'}{M \cdot N \sim M' \cdot N'}$$

Then $T(X) = \{X \vdash M\} / (\sim \cup \sim^{t})^{*}$ → reflexive & transitive; → symmetric

Equivalently, $T(X) = X^{*} = \sum_{n \in \mathbb{N}} X^{n}$.

Applying $f : X \to Y$ induces a map

$$\begin{aligned} Tf : X^* = TX &\longrightarrow Y^* = TY \\ (x_1, \dots, x_n) &\longmapsto (f(x_1), \dots, f(x_n)). \end{aligned}$$

Also, $Tf \circ Tg = T(f \circ g)$ and $T\,\mathrm{id}_X = \mathrm{id}_{TX} = \mathrm{id}_{X^*}$

Basic idea of a monad: a set X with a map $a : TX \to X$.

There is an inclusion $X \to T(X)$:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, equal] & TX \arrow[d, "a"] \\ & X \end{tikzcd}$$

& naturality of η and μ

and compositionnality

$$\begin{tikzcd}[column sep=small] TTX \arrow[rrr, "\mu"] \arrow[dddd, "Ta"'] & & & TX \arrow[dddd, "a"] \\ & ((x_1^1, \dots, x_{n_1}^1), \dots, (x_1^k, \dots, x_{n_k}^k)) \arrow[r, mapsto] \arrow[dd, mapsto] & (x_1^1, \dots, x_{n_k}^k) \arrow[d, mapsto] & \\ & & a(x_1, \dots, x_1^k, \dots, x_{n_k}^k) \arrow[d, equal] & \\ & (a(x_1^1, \dots), \dots, a(x_1^k, \dots)) \arrow[r] & a(a(x_1, \dots), \dots, a(x_1^k, \dots)) & \\ TX \arrow[rrr, "a"'] & & & X \end{tikzcd}$$

A T-algebra is a map $a : T(X) \to X$ such that

$$\begin{tikzcd} TTX \arrow[r, "\mu"] \arrow[d, "Ta"'] & TX \arrow[d, "a"] \\ TX \arrow[r, "a"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, equal] & TX \arrow[d, "a"] \\ & X \end{tikzcd}$$

There is thus a category T-$\mathcal{A}lg$: objects are T-algebras

$f : X \to Y$ induces
$$\begin{tikzcd} TX \arrow[r, "Tf"] \arrow[d, "a_X"'] & TY \arrow[d, "a_Y"] \\ X \arrow[r] & Y \end{tikzcd}$$

When T is the free monoid functor, then $\mathcal{M}on \cong T$-$\mathcal{A}lg$.

$$\begin{tikzcd} \mathcal{M}on \arrow[rr, shift left, "L"] \arrow[dr, "U"'] & & T\text{-}\mathcal{A}lg \arrow[ll, shift left, "R"] \arrow[dl, "V"] \\ & \mathcal{S}et & \end{tikzcd}$$

For R, take $1 := a()$
take $x \cdot y := a(x,y)$

We have
$a(x, a()) = a(a(x), a()) \overset{(\mu)}{=} a(x) = x$

For L, define $a(x_1, \dots, x_n) := ((x_1 \cdot x_2) \cdot x_3 \cdots) \cdot x_n$.

Then we have to extend this map of objects to a functor

$L : \mathcal{M}on \to T$-$\mathcal{A}lg$
$(M, \cdot, 1) \mapsto (M, (x_1, \dots, x_n) \mapsto (x_1 \cdot x_2) \cdot \cdots x_n)$
$f : (M, *, 1') \to (N, \cdot, 1) \mapsto Lf : (M, \dots) \to (N, \dots)$
$x \mapsto f(x)$

and $$\begin{aligned} R : 1\text{-}\mathcal{A}lg &\longrightarrow \mathcal{M}on \\ (M, a) &\longmapsto (M, a(-,-), a()) \\ f : (M,a) \to (N, a') &\longmapsto Rf : (M, a(-,-), a()) \to (N, a'(-,-), a'()) \\ & \qquad\qquad\qquad\quad x \longmapsto f(x) \end{aligned}$$

Indeed,

$$Rf(\underbrace{a(x,y)}_{x \cdot y}) = f(a(x,y)) = a'(f(x), f(y)) = Rfx \cdot Rfy.$$

and $Rf(1) = f(a()) = a()$.

Also,

$$\begin{tikzcd} TLM \arrow[r, "TLf"] \arrow[d, "(*)^n"'] & TLN \arrow[d, "(\cdot)^n"] \\ LM \arrow[r, "Lf"'] & LN \end{tikzcd}$$

by induction on n (as case $n=1$ is verified with $f(1) = 1$ and case $n > 1$ is verified with $f(x*y) = f(x) \cdot f(y)$

A morphism $f : X \to Y$ in ...

$\mathcal{M}on$

$$\begin{tikzcd} X \times X \arrow[r, "f \times f"] \arrow[d, "- * -"'] \arrow[dr, phantom, "f(x*y) = f(x) \cdot f(y)"] & Y \times Y \arrow[d, "- \cdot -"] \\ X \arrow[r, "f"] & Y \\ & 1 \arrow[ul, "1_X"] \arrow[u, "1_Y"'] \end{tikzcd}$$

$1\text{-}\mathcal{A}lg$

$$\begin{tikzcd} TX \arrow[r, "Tf"] \arrow[d, "a"'] & TY \arrow[d, "a'"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

A monad in a category $\mathcal{C}$ is a endofunctor $T: \mathcal{C} \to \mathcal{C}$ with

- unit $\eta_X : X \to TX$ and • multiplication $\mu_X : TTX \to TX$

such that $\eta : id \Rightarrow T$ and $\mu : TT \Rightarrow T$ are natural transformations, such that both diagrams commute.

$$\begin{tikzcd}
TX \arrow[dr, equal] & TTX \arrow[l, "\eta_{TX}"'] \arrow[r, "T\eta_X"] \arrow[d, "\mu_X"] & TX \arrow[dl, equal] \\
& TX &
\end{tikzcd}
\qquad
\begin{tikzcd}
TTTX \arrow[r, "T\mu_X"] \arrow[d, "\mu_{TX}"'] \arrow[dr, phantom, "\text{associativity}"] & TTX \arrow[d, "\mu_X"] \\
TTX \arrow[r, "\mu_X"] & TX
\end{tikzcd}$$

A T-algebra is an object $A \in \mathcal{C}$ and $a : TX \to X$ such that

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, equal] & TX \arrow[dl, "a"] \\
X &
\end{tikzcd}
\qquad
\begin{tikzcd}
FFX \arrow[r, "\mu_X"] \arrow[d, "Fa"'] & FX \arrow[d, "a"] \\
FX \arrow[r, "a"'] & X
\end{tikzcd}$$

A natural transformation $\alpha : F \Rightarrow G$ where $F, G : \mathcal{C} \to \mathcal{D}$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & \mathcal{D}
\arrow[from=U, to=D, Rightarrow, "\alpha"]
\end{tikzcd}$$

is a set of morphisms $\alpha_X : FX \to GX$ such that, for every $f : X \to Y$, the square

$$\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, "\alpha_X"'] & FY \arrow[d, "\alpha_Y"] \\
GX \arrow[r, "Gf"'] & GY
\end{tikzcd}$$

commutes.

Exercise Find T on $\mathcal{G}raph$ such that $\mathcal{C}at \cong T\text{-}\mathcal{A}lg$.

$$
\begin{aligned}
T : \mathcal{G}raph &\to \mathcal{G}raph \\
G &\mapsto G^* \\
f : G \to H &\mapsto f^* : G^* \to H^*
\end{aligned}
$$

Then, a T-algebra is a map $G^* \to G$ in $\mathcal{G}raph$.

$$
\begin{tikzcd}
\mathcal{C}at \arrow[rr, bend left, "L"] \arrow[dr] & & T\text{-}\mathcal{A}lg \arrow[ll, bend left, "R"] \arrow[dl] \\
& \mathcal{S}et &
\end{tikzcd}
$$

$$
\begin{aligned}
R : T\text{-}\mathcal{A}lg &\longrightarrow \mathcal{C}at \\
(G, a : G^* \to G) &\longmapsto \left(\begin{array}{ll} \text{obj}: & \text{vertices of } G \\ \text{morphisms}: & \text{paths in } G \\ \text{composition}: & p \circ q = a(p \cdot q) \\ \text{identity}: & id = a() \end{array}\right) \\
\big(f : (G,a) \to (H,b)\big) &\longmapsto Rf : \begin{array}{l} g \mapsto f(g) \\ g \xrightarrow{*} h \mapsto f(g) \xrightarrow{*} f(h) \end{array}
\end{aligned}
$$

and

$$
\begin{aligned}
L : \mathcal{C}at &\longrightarrow T\text{-}\mathcal{A}lg \\
\mathcal{C} &\longmapsto \big(|\mathcal{C}|,\ a : (x_1, \dots, x_n) \mapsto x_1 \circ \dots \circ x_n\big) \\
F : \mathcal{C} \to \mathcal{D} &\longmapsto LF : x \mapsto F(x)
\end{aligned}
$$

$$
\begin{tikzcd}
G \arrow[r, "\eta_G"] \arrow[dr, equal] & G^* \arrow[d, "a"] \\
& G
\end{tikzcd}
\qquad
\begin{tikzcd}
G^{**} \arrow[r, "\text{concatenation}"] \arrow[d, "a^*"'] & G^* \arrow[d, "a"] \\
G^* \arrow[r, "a"'] & G
\end{tikzcd}
$$

Lecture #2 25/11/2025.

Last-time: products, monads, algebras, finally equivalence between 1-Alg and some other categories

"For any monad T, the forgetful functor $1\text{-Alg} \to \mathscr{C}$ creates products"
$(X,a) \longmapsto X$

We say $U : \mathcal{E} \to \mathcal{B}$ creates binary products if

any product $UX \leftarrow P \to UY$ in $\mathcal{B}$ has a unique antecedant $X \leftarrow \bar{P} \to Y$ in $\mathcal{E}$ which is also a product in $\mathcal{E}$.

This is different from product preservation (U product = product), or product reflexion (if $U(X \leftarrow P \to Y)$ is a product then so is $X \leftarrow P \to Y$).

Proposition: For any monad $T : \mathscr{C} \to \mathscr{C}$, the forgetful functor T-Alg creates binary products.

Let $TA \xrightarrow{r} A$ and $TB \xrightarrow{s} B$ in T-Alg.

We want a T-algebra $T(A \times B) \to A \times B$.

$$\begin{tikzcd}
TA & T(A\times B) \arrow[l, "T\pi_1"'] \arrow[r, "T\pi_2"] \arrow[d, dashed, "\exists!"] & TB \arrow[d, "s"] \\
A & A\times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B
\arrow[from=1-1, to=2-1, "t"']
\end{tikzcd}$$

We need to check the monad laws apply correctly to $T(A\times B)$.

$$\begin{tikzcd}
TT(A\times B) \arrow[rr, "Tf"] \arrow[dr, "TT\pi_1"] \arrow[dd, "\mu_{A\times B}"'] \arrow[drr, phantom, "T\text{déf.}"] & & T(A\times B) \arrow[dr, "T\pi_1"] \arrow[dd] & \\
 & TTA \arrow[rr, "Tr"] \arrow[dd, "\mu_A"] \arrow[drr, phantom, "\text{déf.}"] & & TA \arrow[dd, "r"] \\
T(A\times B) \arrow[rr, "Tf"] \arrow[dr, "T\pi_1"'] \arrow[ur, phantom, "\text{naturality}"] \arrow[r, phantom, "\text{déf}"] & & A\times B \arrow[dr] \arrow[l, phantom, "\text{déf.}"] & \\
 & TA \arrow[rr, "r"'] & & A
\end{tikzcd}$$

So we have compositionnality.

$$
\begin{tikzcd}
A \arrow[dd, phantom, ""] \arrow[d, "\eta_A"'] \arrow[dr, phantom, "\text{naturality}"] & A\times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\eta_{A\times B}"] \arrow[dr, phantom, "\text{nat.}"] & B \arrow[d, "\eta_B"] \\
TA \arrow[d, "r"'] \arrow[dr, phantom, "\text{def.}"] & T(A\times B) \arrow[l, "T\pi_1"] \arrow[r, "T\pi_2"'] \arrow[d, "f"] \arrow[dr, phantom, "\text{def}"] & TB \arrow[d, "t"] \\
A & A\times B \arrow[l, "\pi_1"] \arrow[r] & B
\end{tikzcd}
$$

It remains to show that $T(A\times B)$ is a product in T-Alg.

$$
\begin{tikzcd}
& TP \arrow[ddl, "Tg"'] \arrow[d, "s"] \arrow[dd, bend left, "Tp"] \arrow[ddr, "Th"] & \\
& P \arrow[ddl, "g"'] \arrow[dd, bend right, "\exists!\, p"'] \arrow[ddr, "h"] & \\
TA \arrow[d, "r"'] \arrow[dr, phantom, "\text{def}"] & T(A\times B) \arrow[l, "T\pi_1"] \arrow[r, "T\pi_2"] \arrow[d, "f"] \arrow[dr, phantom, "\text{def}"] & B \arrow[d, "t"] \\
A & A\times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & TB
\end{tikzcd}
$$

(This diagram is in $\mathscr{C}$ but talks about T-Alg).

This is where we start with combinatorial constructions.

$\boxed{\text{Pre Sheaves}}$

A presheaf is a functor $\mathscr{C}^{op} \longrightarrow \mathcal{S}et$.

The opposite category $\mathscr{C}^{op}$ has the same objects and morphisms from A to B are $\mathscr{C}(B,A)$ and we use reverse composition in $\mathscr{C}$

$$\begin{tikzcd}
\mathscr{C}^{op}(B,C) \times \mathscr{C}^{op}(A,B) \arrow[d, "\cong"] \arrow[rr, "\circ_{\mathscr{C}^{op}}"] & & \mathscr{C}^{op}(A,C) \arrow[d, equal] \\
\mathscr{C}^{op}(A,B) \times \mathscr{C}^{op}(B,C) \arrow[d, equal] & & \mathscr{C}(C,A) \\
\mathscr{C}(B,A) \times \mathscr{C}(C,B) \arrow[urr, "\circ_{\mathscr{C}}"] & &
\end{tikzcd}$$

Define $\mathcal{G}$: $\begin{tikzcd}[0] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & [1]\end{tikzcd}$. Presheaves F on $\mathcal{G}$ are exactly directed multigraphs

$$\begin{tikzcd}
\underbrace{F[0]}_{V} & \underbrace{F[1]}_{E} \arrow[l, bend right, "Fs"'] \arrow[l, bend left, "Ft"]
\end{tikzcd}$$

Define $PSh(\mathscr{C}) = \hat{\mathscr{C}} = [\mathscr{C}^{op}, Set]$. So $\mathcal{G}ph := \hat{\mathcal{G}}$.

((Remark $\mathcal{G}$ is small but $\hat{\mathcal{G}}$ is only locally small.

Graph morphisms should make

$$\begin{tikzcd}
F[1] \arrow[r, "f_{[1]}"] \arrow[d, "Fs"'] & G[1] \arrow[d, "Gs"] \\
F[0] \arrow[r, "f_{[0]}"'] & G[0]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
F[1] \arrow[r, "f_{[1]}"] \arrow[d, "Ft"'] & G[1] \arrow[d, "Gt"] \\
F[0] \arrow[r, "f_{[0]}"'] & G[0]
\end{tikzcd}$$

commute, which we will write as

$$\begin{tikzcd}
F[1] \arrow[r, "f_{[1]}"] \arrow[d, shift right, "Fs"'] \arrow[d, shift left, "Ft"] & G[1] \arrow[d, shift right, "Gs"'] \arrow[d, shift left, "Gt"] \\
F[0] \arrow[r, "f_{[0]}"'] & G[0]
\end{tikzcd}$$

should <u>serially</u> commute.

A <u>coproduct</u> is of the form

$$\begin{tikzcd}
X \arrow[r, "\iota_1"] \arrow[dr, "f"'] & X+Y \arrow[d, "\exists!"] & Y \arrow[l, "\iota_2"'] \arrow[dl, "g"] \\
& A &
\end{tikzcd}$$

In Set, it can be constructed as $\{0\} \times X \sqcup \{1\} \times Y$.

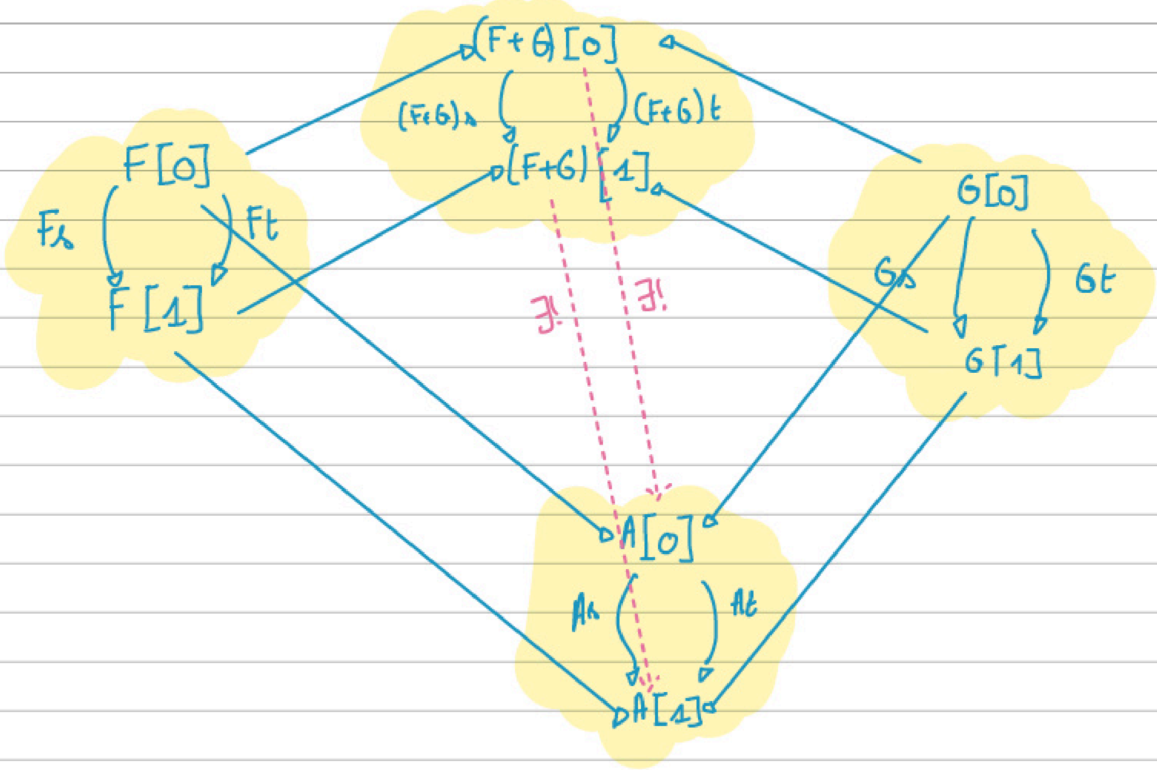
How to compute a coproduct of graphs?

$$\begin{tikzcd}
& (F+G)[0] \arrow[d, shift right, "(F+G)s"'] \arrow[d, shift left, "(F+G)t"] & \\
F[0] \arrow[ur] \arrow[d, shift right, "Fs"'] \arrow[d, shift left, "Ft"] & (F+G)[1] & G[0] \arrow[ul] \arrow[d, shift right, "Gs"'] \arrow[d, shift left, "Gt"] \\
F[1] \arrow[ur] & & G[1] \arrow[ul]
\end{tikzcd}$$

should sequentially commute. So we should have

that
$$\begin{aligned}
(F+G)[0] &= F[0] \sqcup G[0] \\
(F+G)[1] &= F[1] \sqcup G[1] \\
(F+G)\,s &= Fs + Gs \\
(F+G)\,t &= Ft + Gt
\end{aligned}$$

Also



Let us talk about colimits. Colimits are the universal cocones.

A diagram in $\mathcal{C}$ is a functor $F : \mathbb{I} \to \mathcal{C}$ for some small $\mathbb{I}$.

A cocone "$\lambda : D \to c$" is a natural transformation from D to the constant functor $c \in \mathcal{C}$.

A cocone morphism is of the form

$$\begin{tikzcd} & D \arrow[dl, "\lambda"'] \arrow[dr, "\lambda'"] & \\ c \arrow[rr, "k"'] & & c' \end{tikzcd}$$

where $k \in \mathcal{C}(c, c')$.

We thus define a category $\mathrm{Cocone}(D)$ of cocones over D.

A colimit of D is the initial object of $\mathrm{Cocone}(D)$:

$\forall c$,

$$\begin{tikzcd} & D \arrow[dl, "\lambda"'] \arrow[dr] & \\ \mathrm{colim}\, D \arrow[rr, "\exists!"'] & & c \end{tikzcd}$$

$\mathcal{2} := \mathcal{1} + \mathcal{1}$

Example: for coproduct, use the diagram $\begin{tikzcd} \bullet \arrow[loop below, "id"] & \bullet \arrow[loop below, "id"] \end{tikzcd}$

for initial object, use the empty diagram .

Indeed, the universal property of colim(empty diagram) tells us that

$$\forall c, \qquad \mathrm{colim}(\emptyset) \xrightarrow{\exists !} c$$

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "i_2"] \arrow[ddr, bend left, "v"] & \\
A \arrow[r, "i_1"'] \arrow[drr, bend right, "u"'] & A +_c B \arrow[dr, dashed, "\overline{\varphi}"] & \\
& & D
\end{tikzcd}$$

$A +_c B := A + B \,/\, f(c) = g(c)$.

$$\begin{cases} i_1 : A \longrightarrow A +_c B \\ i_2 : B \longrightarrow A +_c B \end{cases}$$

$$\begin{aligned} \varphi : A + B &\longrightarrow D \\ a &\longmapsto u(a) \\ b &\longmapsto v(b) \end{aligned}$$

and $\overline{\varphi} : A +_c B \longrightarrow D$
exists as $\varphi(f(c)) = \varphi(g(c))$
$u(f(c)) = v(g(c))$

$$\begin{tikzcd}
\mathcal{2} \arrow[r, "i_0"] \arrow[d, "!"'] & \mathcal{2} + \mathcal{3} \arrow[d] \\
\mathcal{1} \arrow[r] & \mathcal{1} + (\mathcal{2} + \mathcal{3}) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

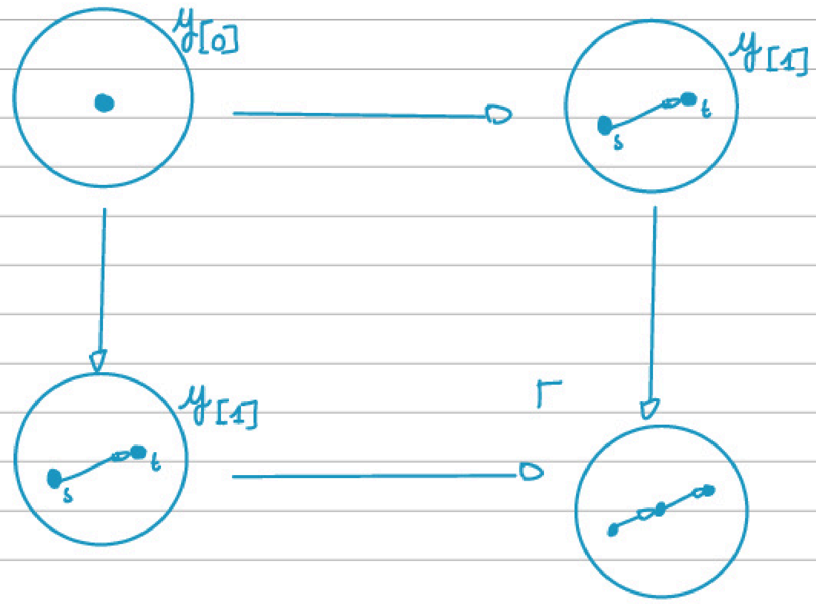
$\mathcal{1} + (\mathcal{2} + \mathcal{3}) \,/\, i_0(0_{\mathcal{2}}) = !_0(0_{\mathcal{2}}) = !_0(1_{\mathcal{2}}) = i_0(1_{\mathcal{2}})$

$\|$

$\mathcal{1} + \mathcal{3} = \mathcal{4}$

The pushout of $\underset{\mathcal{y}[0]}{\overset{\bullet}{\shortparallel}}$, $\underset{\mathcal{y}[1]}{\overset{\bullet_s \to \bullet_t}{\shortparallel}}$ in Graphs along $\mathcal{y}_t : \bullet \mapsto \bullet_t$
$\mathcal{y}_s : \bullet \mapsto \bullet_s$

is $\bullet \to \bullet \to \bullet$



Lecture #3

02/12/2025

Exercise In Set, the equalizer of $A \overset{f}{\underset{g}{\rightrightarrows}} B$ is

$$E := \{ a \in A \mid f(a) = g(a) \}$$

with $i : E \to A$ the inclusion arrow. For any $x : X \to A$ such that $x \circ f = x \circ g$, then $x(X) \subseteq E$ and so there is an unique inclusion arrow

$$\begin{aligned} m : X &\longrightarrow E \\ u &\longmapsto x(u) \end{aligned}$$

such that

$$\begin{tikzcd} X \arrow[d, dashed, "\exists!\, m"'] \arrow[dr, "x"] & & \\ E \arrow[r, "i"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

composes.

<u>Exercise</u> • $[\mathcal{C}, \mathcal{D}]$ is a category $\big(F \circ G(X) = F(G(X))$
$F \circ G(f) = F(G(f))$
$Id(X) = X \quad Id(f) = f \big)$

• $ob([\mathcal{C}, \mathcal{D}]) \subseteq ob(\mathcal{D})^{ob(\mathcal{C})} \times Mor(\mathcal{D})^{Mor(\mathcal{C})}$ and so $ob([\mathcal{C}, \mathcal{D}])$ is a set as $\mathcal{C}$ and $\mathcal{D}$ are small.

For some $F, G \in [\mathcal{C}, \mathcal{D}]$, $\quad Hom(F, G) \subseteq \prod_{X \in \mathcal{C}} Hom^{\mathcal{D}}(FX, GX)$ is a set.
↳ small

• For some $F, G \in [\mathcal{C}, \mathcal{D}]$, $\quad Hom(F, G) \subseteq \prod_{X \in \mathcal{C}} Hom(FX, GX)$ is a set.
↳ in $\mathcal{D}$ locally small
↳ small

• The functor $F : [\mathcal{C}, \mathcal{D}] \longrightarrow [ob(\mathcal{C}), \mathcal{D}]$ is defined as

$$(U : \mathcal{C} \to \mathcal{D}) \longmapsto \bar{U} : X \longmapsto UX$$
$$\bar{U} : id_X \longmapsto id_{UX}$$
$$(\eta : U \Rightarrow V) \longmapsto \bar{\eta} : \bar{U} \Longrightarrow \bar{V}$$
$$\bar{U}X = UX \longrightarrow VX = \bar{V}X$$

$$\begin{tikzcd} UX \arrow[r, "\eta_X"] \arrow[d, "Uf"'] & VX \arrow[d, "Vf"] \\ UY \arrow[r, "\eta_Y"'] & VY \end{tikzcd}$$

Take a product

$$FA \overset{\pi_1}{\Longleftarrow} X \overset{\pi_2}{\Longrightarrow} FB$$

in $[ob(\mathcal{C}), \mathcal{D}]$, then

$$A = GFA \overset{G\pi_1}{\Longleftarrow} GX \overset{G\pi_2}{\Longrightarrow} GFB = B$$

with $G : [ob(\mathcal{C}), \mathcal{D}] \longrightarrow [\mathcal{C}, \mathcal{D}]$ the inclusion map.

Take $P \in [\mathcal{C}, \mathcal{D}]$ with $\eta : P \Rightarrow A$ $\mu : P \Rightarrow B$ commutes

$$\begin{tikzcd} & P \arrow[dl, Rightarrow, "\eta"'] \arrow[dr, Rightarrow, "\mu"] & \\ A & GX \arrow[l, Rightarrow, "G\pi_1"] \arrow[r, Rightarrow, "G\pi_1"'] & B \end{tikzcd}$$

in $[\mathcal{C}, \mathcal{D}]$

Then we have that

$$\begin{tikzcd} & FB \arrow[dl, Rightarrow, "F\eta"'] \arrow[d, Rightarrow, dotted, "\gamma"] \arrow[dr, Rightarrow, "F\mu"] & \\ FA & X \arrow[l, Rightarrow, "\pi_1"] \arrow[r, Rightarrow, "\pi_1"'] & FB \end{tikzcd}$$

commutes in $[ob(\mathcal{C}), \mathcal{D}]$ and so

$$\begin{tikzcd} & P \arrow[dl, Rightarrow, "\eta"'] \arrow[d, Rightarrow, "G\gamma"] \arrow[dr, Rightarrow, "\mu"] & \\ A & GX \arrow[l, Rightarrow, "G\pi_1"] \arrow[r, Rightarrow, "G\pi_1"'] & B \end{tikzcd} \quad \text{in } [\mathcal{C}, \mathcal{D}]$$

commutes. Unicity is by $Nat(P, GX) \hookrightarrow Nat(FP, X)$.

Given a graph G, we define $el(G)$ the category

- <u>objects</u> pairs (c, x) with $c \in \mathcal{G}$ and $x \in X(c)$
- <u>morphisms</u> $f : (c, x) \longrightarrow (c', y)$ st $f : c \to d$ and $X(f)(y) = x$.

$$el(X) \xrightarrow{\pi_X} \mathcal{G} \xrightarrow{y} \hat{\mathcal{G}} = \mathcal{G}raph.$$

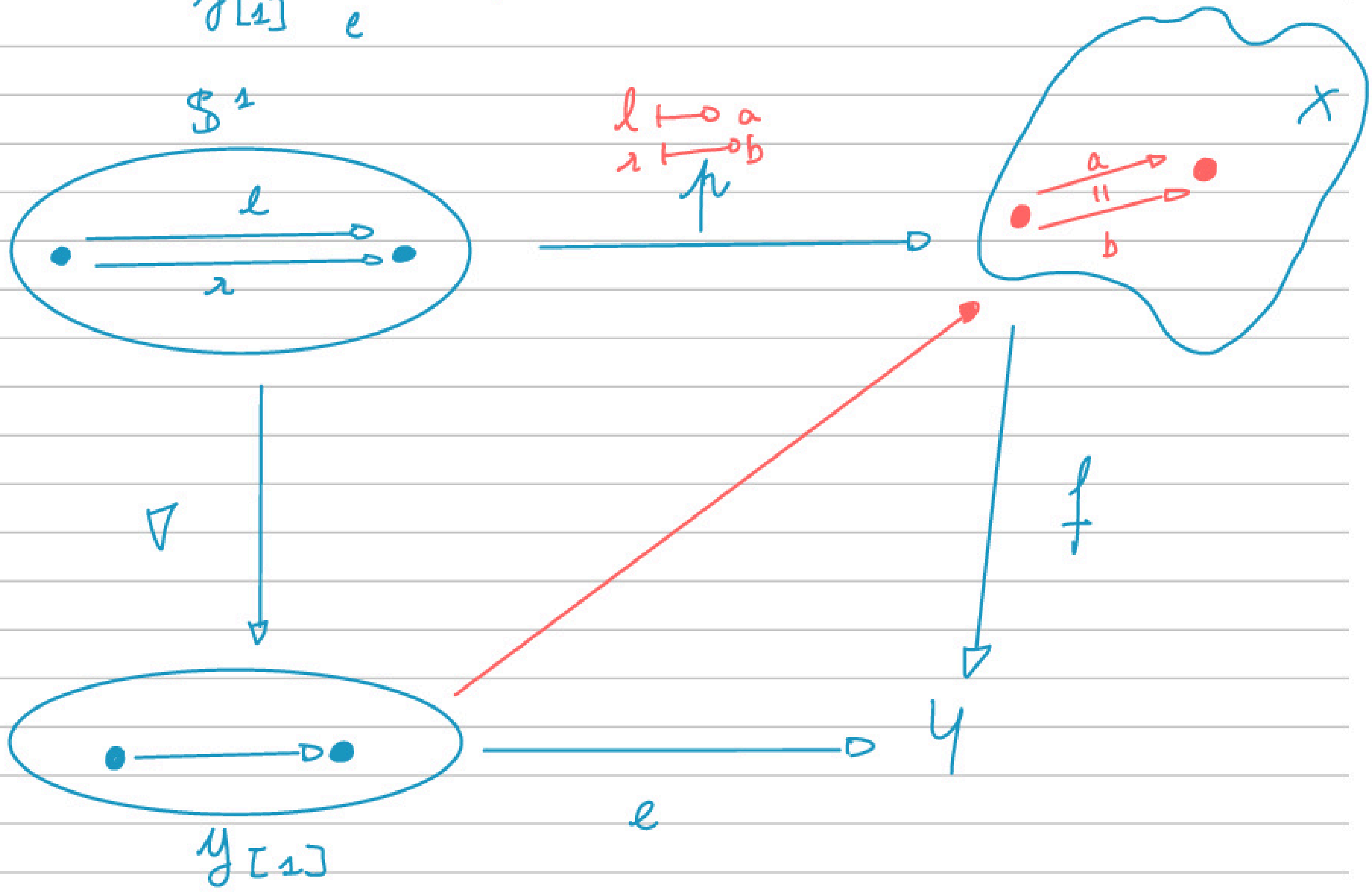
<u>Proposition</u> (co-Yonneda) Every presheaf $X \in \hat{\mathcal{C}}$ is the colimit of

$$el(X) \xrightarrow{\pi_X} \mathcal{C} \longrightarrow \hat{\mathcal{C}}$$

for a small $\mathcal{C}$.

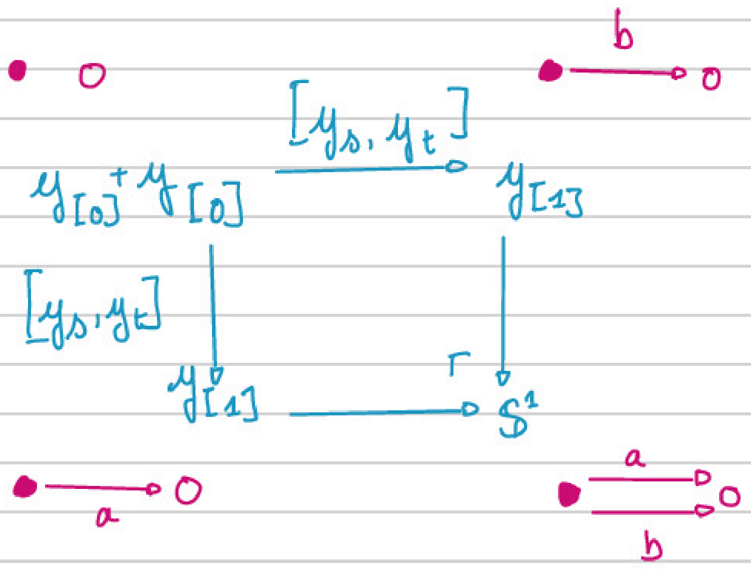
A functor $F : \mathcal{C} \to \mathcal{D}$ is *faithful* if, for every $F_{AB} : \mathcal{C}(A,B) \longrightarrow \mathcal{D}(A,B)$ is injective.

$$\begin{tikzcd}
\mathbb{S}^1 \arrow[r, "p"] \arrow[d, "\nabla"'] & X \arrow[d, "f"] \\
y_{[1]} \arrow[r, "e"'] \arrow[ur, dotted] & Y
\end{tikzcd}$$



<u>Exercise</u>

$$\begin{tikzcd}
y_{[0]} + y_{[0]} \arrow[r, "{[y_s, y_t]}"] \arrow[d, "{[y_s, y_t]}"'] & y_{[1]} \arrow[d] \\
y_{[1]} \arrow[r] & \mathbb{S}^1 \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$



A graph is simple if $\bullet \overset{e}{\underset{e'}{\rightrightarrows}} \bullet$ implies $e = e'$

Equivalently if $X \longrightarrow T$ is faithful.

Surjective / Injective functions in Set form a factorization system

$$\begin{tikzcd}
1 \arrow[r, "a"] \arrow[dr, "f(a)"'] \arrow[r, phantom, shift right=6, "\circlearrowright"] & A \arrow[r, "s"] \arrow[d, "f"] \arrow[dr, phantom, shift left=3, "*"] & X \arrow[d, "\iota_1"] \\
& B \arrow[r, "h"'] \arrow[ur, "f(a) \mapsto s(a)"] \arrow[ur, phantom, shift right=3, "**"] & X+Y
\end{tikzcd}$$

We have that

$$\begin{tikzcd}
A \arrow[r, "s"] \arrow[d, "f"'] & X \arrow[d, "\iota_1"] \\
B \arrow[r, "h"'] & X+Y
\end{tikzcd}$$

commutes, and so we immedially have the required commutations $(*)$ and $(**)$.

Injective / surjective too

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\iota_1"'] \arrow[dr, phantom, shift left=3, "*"] & A+B \arrow[d, "{[A,k]}"] \\
X+Y \arrow[r, "{[g,h]}"'] \arrow[ur, "{[f,\iota_1 k]}"'] \arrow[r, phantom, shift left=2, "**"] & A
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\text{bijective on objects}"] \arrow[rr, bend right, "F"'] & M_F \arrow[r, "\text{fully faithful}"] & \mathcal{D}
\end{tikzcd}$$

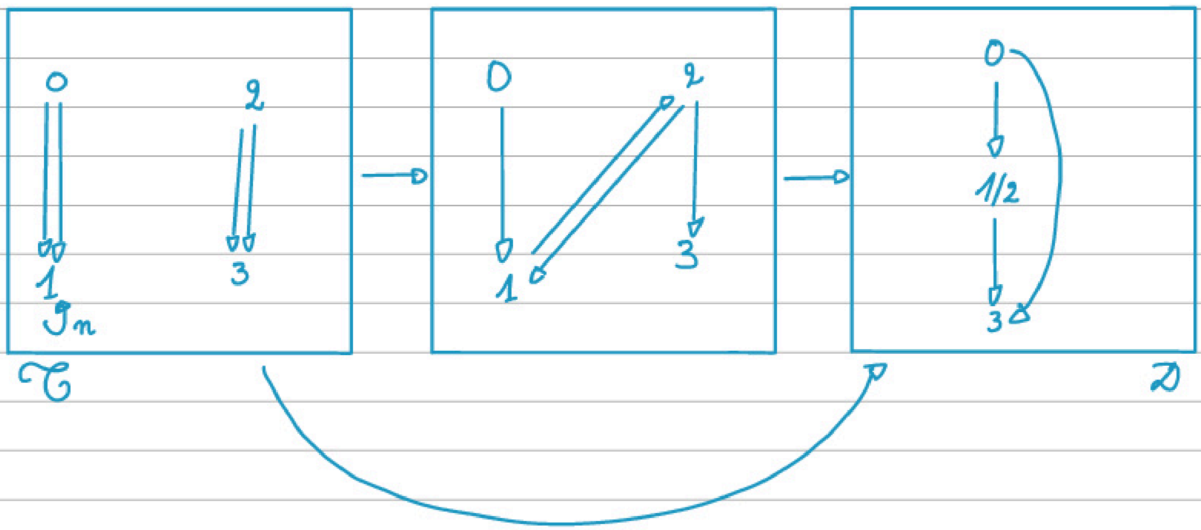
M_F : objects: $ob(\mathcal{C})$

morphisms:

$\mathcal{C}(A,B) / \{f, g \mid Ff = Fg\}$.

or alternatively,

$\mathrm{Hom}_{M_F}(A,B) := \mathrm{Hom}_{\mathcal{D}}(FA, FB)$



Theorem. $({}^{\pitchfork}(\mathcal{J}^{\pitchfork}), {}^{\pitchfork}\mathcal{J})$ is a factorization system where $\mathcal{J}$ is a set of morphisms in $\mathcal{T}$ (for some "suitable" $\mathcal{T}$).

Exercise Design a $\mathcal{J}$ such that the full subcategory of graphs X with $!_X \in \mathcal{J}^{\pitchfork}$ is equivalent to the category of undirected graph.

$$\mathcal{J} := \left\{ \begin{array}{l} \nabla : \mathbb{S}^1 \longrightarrow y_{[1]} \\ U : y_{[1]} \longrightarrow \mathbb{S}^1_{\circlearrowright} \end{array} \right\}$$

Where $\mathbb{S}^1_{\circlearrowright} : \bullet \rightleftarrows \bullet$

$$\begin{tikzcd}
\omega \arrow[rr, "y"] \arrow[dr, bend right, "R"'] & \arrow[d, phantom, "\Downarrow"] & \hat{\omega} \arrow[dl, bend left, "R^*"] \\
& \mathcal{G}raph &
\end{tikzcd}$$

$R(n) := 0 \to 1 \to \cdots \to n-1$

$R(n \leq p) :=$ prefix $R(n) \to R(p)$

$R^*(X)(n) = \mathcal{G}raph(R(n), X)$ = paths of length n

$R^*(X)(n \leq p)$: inclusion to length p

Comma category

Given $\mathcal{C} \xrightarrow{F} \mathcal{E} \xleftarrow{G} \mathcal{D}$

define F/G with

- <u>objects</u>: $(C \in \mathcal{C}, D \in \mathcal{D}, f \in \mathcal{E}(FC, GD))$
- <u>morphisms</u>: $(C, D, f) \xrightarrow{(u,v)} (C', D', f')$ such that

$$\begin{tikzcd}
FC \arrow[r, "u"] \arrow[d, "f"'] & FC' \arrow[d, "g"] \\
GD \arrow[r, "v"'] & GD'
\end{tikzcd}$$

This forms a category:

$$\begin{tikzcd}
FC \arrow[r, equal] \arrow[d, "f"'] & FC \arrow[d, "f"] \\
GD \arrow[r, equal] & GD
\end{tikzcd}
\qquad
\begin{tikzcd}
FC \arrow[r, "u"] \arrow[d, "f"'] \arrow[rr, bend left, "u' \circ u"] & FC' \arrow[r, "u'"] \arrow[d, "g"] & FC'' \arrow[d, "h"] \\
GD \arrow[r, "v"'] \arrow[rr, bend right, "v' \circ v"'] & GD' \arrow[r, "v'"'] & GD''
\end{tikzcd}$$

$$\begin{tikzcd} \mathcal{G}raph \arrow[r] & \mathcal{S}et & \mathbb{1} \arrow[l, "\mathbb{A}"'] \\[-20pt] G \arrow[r, mapsto] & \mathbb{E}(G) = G[1] & \end{tikzcd}$$

Then the comma category $\mathbb{E}/\mathbb{A}$ is equivalent to the category of labelled graphs.

$$\begin{tikzcd} X[1] \arrow[rr, "f[1]"] \arrow[dr] & & Y[1] \arrow[dl] \\ & \mathbb{A} & \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{A}^* \arrow[rr] \arrow[dr, "R"'] & & \widehat{\mathbb{A}^*} \\ & \mathcal{G}raph/\mathbb{A} \arrow[ur, "R^*"'] & \end{tikzcd}$$

$$R^*(X,\ell)(a_1 \dots a_n) = \mathcal{G}raph/_{\mathbb{A}}\big(([n+1],[a_1 \dots a_n]),(X,\ell)\big).$$